SF2A 2006
D. Barret, F. Casoli, T. Contini, G. Lagache, A. Lecavelier, and L. Pagani (eds)

A PRESCRIPTION FOR THE SOFTENING LENGTH IN FLAT DISKS

J.-M. Huré[1,2] and A. Pierens[3]

Abstract. Whereas "soften gravity" simplifies the computation of the gravitational potential due to continuous distributions of matter, it usually fails to give reliable values. For flat disks, λ is set to a certain fraction of the (real) disk thickness. However, this fraction seems arbitrary and varies from one author to the other. We have determined this fraction for axisymmetrical systems from a rigorous mathematical analysis.

1 Introduction

Newton's generalized law for gravity exhibits a divergence as the separation $|\vec{r}-\vec{r'}|$ between any continuous mass element and the field point vanishes, although volume integrals are finite. In practice, this difficulty is avoided by considering a parameter λ (the "softening" length) such that any separation writes $|\vec{r}-\vec{r'}| \approx \sqrt{|\vec{r}-\vec{r'}|^2+\lambda^2} \neq 0$ close to the singularity. For continuous flat disks, people usually set λ to some fraction of the disk thickness arguing that λ mimics the vertical disk extension, which is certainly is correct, *qualitatively*. However, *from a quantitative point of view*, one can not expect potential and field values better than 10% in relative using a constant value for λ as commonly done (or any ad-hoc function). It is worth noting that "soften gravity" with an inappropriate λ-value makes i) gravity significantly non-Newtonian and ii) models and simulations strongly λ-dependent. Here, we propose the first reliable prescription for λ ever derived for axi-symetrical systems.

2 Potential due to an axi-symmetrical torus in terms of elliptic integrals

The potential due to any axially symmetric torus reads (e.g., Durand 1964):

$$\Psi(\vec{r}) = -2G \iint_{a,z} \sqrt{\frac{a}{R}} \rho k \mathbf{K}(k) da dz, \qquad k = \frac{2\sqrt{aR}}{\sqrt{(a+R)^2+(z-Z)^2}}, \tag{2.1}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind, k is the modulus, (a, z) and (R, Z) are the cylindrical coordinates of source points and field point respectively, and ρ is the mass density. To get a reliable representation of Ψ near singularities, we have considered an expansion of $\mathbf{K}(k)$ over the complementary modulus k' (e.g. Gradshteyn & Ryzhik 1994):

$$\mathbf{K}(k) = \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2(n+1)!!} P_n(k') {k'}^{2n}, \qquad k' = \sqrt{1-k^2}, \qquad P_{n+1}(k') = P_n(k') - \frac{1}{(2n+1)(n+1)}, \tag{2.2}$$

with $P_0 = \ln \frac{4}{k'}$. Very close to the singularity where $k' \to 0$, the potential is remarkably well described by a single term, namely P_0. Interestingly enough, the integration of the Poisson kernel along the z-direction is possible analytically by setting $\mathbf{K}(k) = P_0$ (second order accurate in k') in Eq.(2.1a), provided ρ remains a "simple" function of the altitude. In particular, if the torus is vertically homogeneous (i.e. ρ depends on a only), we can derive for Ψ an expression resembling Eq.(2.1a), namely

$$\Psi(\vec{r}) \approx -2G \int_a \sqrt{\frac{a}{R}} \Sigma(a) k_\lambda \mathbf{K}(k_\lambda) da, \qquad k_\lambda = \frac{2\sqrt{aR}}{\sqrt{(a+R)^2+\lambda^2}}, \qquad \Sigma = \int_z \rho dz = 2\rho h, \tag{2.3}$$

where $2h$ is the local thickness of the torus, and λ is a function of a, h, R and Z (Pierens & Huré 2007).

[1] Université Bordeaux 1 et L3AB/Observatoire de Bordeaux, 2 rue de l'Observatoire, BP 89, 33270 Floirac
[2] LUTh/Observatoire de Paris-Meudon (research associate), Place Jules Janssen, 92195 Meudon Cedex, France
[3] Astronomy Unit, Queen Mary, University of London, Mile end Road, London E1 4NS, UK

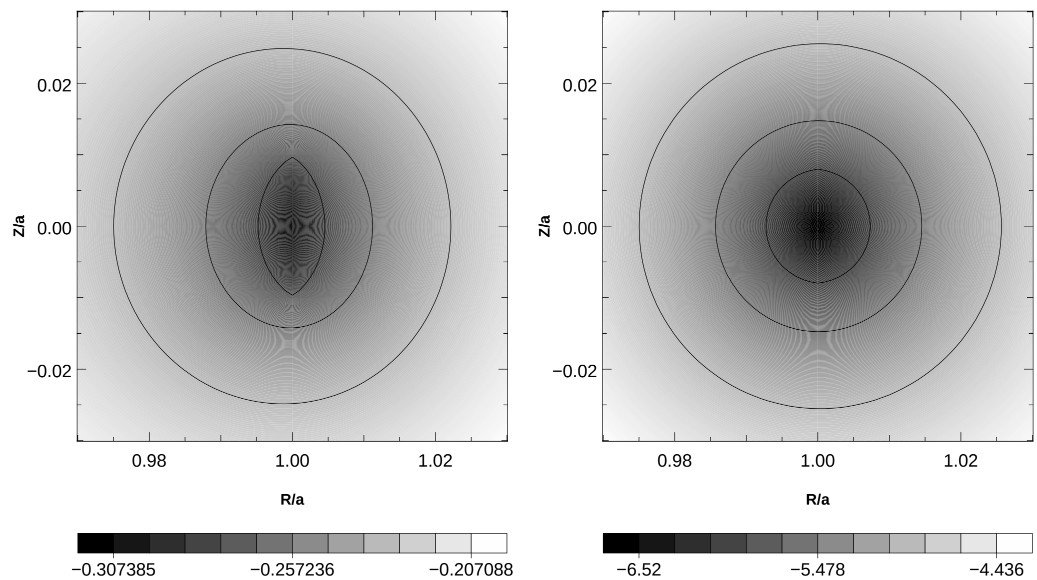

Fig. 1. *Left*: the quantity $\frac{1}{2h}\int_z k\mathbf{K}(k)dz$ resulting from the integration of the Poisson kernel in the z-direction (see Eq.(2.1)). *Right*: decimal logarithm of the relative deviation between $\frac{1}{2h}\int_z k\mathbf{K}(k)dz$ and its approximation, $k_\lambda\mathbf{K}(k_\lambda)$.

3 A prescription for the softening length in toroidal cells

Eq.(2.3a) is especially interesting because it is just the expression for the potential due to a flat (i.e. $h = 0$) disk with surface density Σ, as commonly defined *in the framework of soften gravity* where $|\vec{r}-\vec{r}'| \approx \sqrt{|\vec{r}-\vec{r}'|^2+\lambda^2}$. At the center of the torus (i.e. $Z = 0$) and assuming that the torus is symmetrical with respect to the equatorial plane, the inferred prescription for λ greatly simplifies since we find:

$$\ln\frac{\lambda}{h} \approx -1 + \frac{4}{3}\frac{h^2}{(a+R)^2}. \tag{3.1}$$

As a matter of fact, this calculus assumes $\frac{h}{a+R} \ll 1$ and $a \sim R$ meaning that the torus considered here has necessarily a small spatial extension (i.e. it is rather a "toroidal cell"). Under these conditions, Eq.(3.1) gives $\lambda \approx h/e$. Figure 1a shows the quantity $\frac{1}{2h}\int_z k\mathbf{K}(k)dz$ determined numerically for $h/a = 0.01$ within the computer precision (Huré 2005). This quantity must be compared to our $\mathcal{O}(k'^2)$-approximation, namely $k_\lambda\mathbf{K}(k_\lambda)$. Figure 1b displays the logarithm of the relative deviation between these two quantities (which are both functions of space and torus size). We see that, in the neighborhood of the toroidal cell, soften gravity gives excellent results when used with the "right" prescription for λ (the relative error is $\sim 10^{-6}$ in the present example). The nominal choice for λ as well as a few tests will be reported in Huré & Pierens (2007).

References

Durand, E. 1964, Electrostatique I. Les distributions, Masson Ed.

Gradshteyn, I.S., & Ryzhik, I.M. 1994, Table of integrals, series and products, Acad. Press Inc., 5th ed., A. Jeffrey Ed.

Huré, J.-M. 2005, A&A, 434, 1

Huré, J.M., & Pierens, A. 2007, in preparation