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NUMERICAL STUDY OF STATIONARY BLACK HOLES: LOCAL HORIZON PROPERTIES AND THE KERR SOLUTION

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Abstract. This work focuses on a numerical implementation of the local physics of black hole astrophysical spacetimes. This is done by imposing boundary conditions on a certain formulation of Einstein Equations, namely the fully constrained formalism(FCF) of Bonazzola et al.(2004). We here make use of the Isolated Horizon formalism of Ashtekar et al. (1999), aiming at a local characterization of a black hole region. This horizon can be seen as an intuitive physical object (contrary to, e.g., the event horizon). We thus solve the Einstein Equations, using 3+1 formalism, on 3-slices of spacetime excised by marginally trapped surfaces. We are then able to recover the Kerr spacetime outside the black hole region only by prescribing that our grid boundary behaves indeed like an Isolated Horizon. Contrary to some earlier works, we take into account the non-conformal part of our 3-metric, making use of a no-boundary method on the horizon. Our spacetime is then perfectly stationary.We compare our results with previous ones, and show accuracy results, involving among others a verification of the virial theorem, and a refined Penrose inequality studied in Jaramillo et al. (2007).

1 Introduction

Trying to accurately describe black holes solutions as evolving physical objects in numerical simulations is of direct interest in astrophysics. We focus here on a particular approach, trying to describe black holes as physical objects represented by their horizons, and using numerical excision techniques. Defining the physical laws for event horizons of black holes has been tried some time ago (see Thorne et al.(1986)). However, being global objects, applying evolution laws to event horizons is almost impossible due to their non causal behaviour. Alternative local characterizations, based on the concept of trapped surfaces, have been recently formulated. We will use here the isolated horizon formalism, prescribing the physics of non-evolving black hole regions.

Following the prescription of Gourgoulhon & Jaramillo (2006) and pursuing the numerical explorations of Cook and Pfeiffer (2004) and Jaramillo et al. (2007) among others, we try to numerically implement those objects as boundary conditions imposed on (3+1) Einstein Equations, in a 3-slice excised by a 2-surface. This is done here using the Fully Constrained Formalism (FCF) of Bonazzola et al. (2004), with spectral method high accuracy resolutions using the LORENE library (http://www.lorene.obspm.fr). We drop out the usual conformal flatness hypothesis for our simulation, so that we can exactly recover a stationary rotating vacuum slice of spacetime. Issues raised by this approach require a particular handling of boundary conditions for the non conformally flat equation. In our case, no additional boundary condition is required.

2 Trapped surfaces and Isolated Horizons as a local description of black hole regions

We refer the reader to Ashtekar and Krishnan(2004) for a review. The global concept of a trapped surface relies on the concept of expansion: this is defined as the area rate of change along geodesics orthogonal to a 2-surface in a spacetime (negative if the area is decreasing along the geodesics from the surface, positive otherwise). A spacelike closed 2-surface is said to be a trapped surface if expansions along the two future null geodesics normal to the surface are less than zero. This clearly characterizes strong local curvature. A marginally outer trapped surface in an asymptotically flat spacetime has the expansion along the outer null geodesics normal

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to the surface to be exactly zero. If the cosmic censorship holds, there is an equivalence between a marginally outer trapped surface and the presence of a black hole region enclosing it. These surfaces are always situated inside the event horizon.

Isolated horizons are aimed at describing stationary vacuum black hole regions. They are defined as 3-dimensional tubes foliated by marginally outer trapped surfaces, and with a null vector field as generator. We prescribe in addition that the metric is not evolving along the generators. A consequence is that any slice of an isolated horizon has in addition a vanishing shear tensor. The shear 2-tensor is a geometrical quantity on the 2-surface that measures the geometrical strain undergone by this surface.

This formalism has been extensively studied as a *diagnosis* for simulations involving black holes, where marginally trapped surfaces are found *a posteriori* with numerical tools called apparent horizon finders (Lin et al. 2004). We here want to impose properties on a surface that will turn out to be an Isolated Horizon,and simulate what happens outside of it. This approach has been made by Cook and Pfeiffer (2004) and Jaramillo et al. (2007) for the single black hole case. The main improvement in this work is the dropping out of the conformal flatness hypothesis, leading to more accurately stationary data in the rotating case.

3 A fully constrained formalism of Einstein Equations

For technical details, the reader is referred to Bonazzola et al. (2004) or Gourgoulhon (2007).

The 3+1 formalism of General Relativity, foliates a 4-dimensional globally hyperbolic variety with spatial 3-slices endowed with an induced 3-metric γ_{ij} and an extrinsic curvature K_{ij}. The 4-metric of an asymptotically flat, globally hyperbolic variety ($\mathcal{M}$) writes:$g_{\mu\nu}dx^{\mu}dx^{\nu} = -N^2dt^2 + \psi^{-4}\tilde{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, where N, ψ, β^i and γ_{ij} are respectively the lapse scalar, the conformal factor, the shift vector and the conformal metric. We also make the decomposition $\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$, where f^{ij} is the flat metric associated with our 3-slice, and h^{ij} The deviation of the metric from conformal flatness. Setting arbitrarily $h^{ij} = 0$ before caluclations is called the conformal flatness approximation (or Isenberg-Wilson-Matthews theory of gravity).

This decomposition allows us to write the 3+1 Einstein system (see Gourgoulhon 2007). We will place ourselves in the FCF of Bonazzola et al. (2004). We fix the gauge to be the Dirac gauge: the divergence of the metric with respect to a flat derivative associated with the 3-slice must vanish. Moreover, we seek to find data on a 3-slice that are stationary in our coordinate system (all the time derivatives vanish). Under these approximations, the 3+1 system writes:

$$\Delta\psi = \mathcal{S}_{\psi}(h^{ij}, K^{ij}, N, \psi, \beta^i) \tag{3.1}$$

$$\Delta(N\psi) = \mathcal{S}_{N\psi}(h^{ij}, K^{ij}, N, \psi, \beta^i) \tag{3.2}$$

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^j = (\mathcal{S}_{\beta^k}(h^{kj}, K^{kj}, N, \psi, \beta^k))^i \tag{3.3}$$

$$\Delta h^{ij} - \frac{\psi^4}{N^2}\mathcal{L}_{\beta}\mathcal{L}_{\beta}h^{ij} = \mathcal{S}^{ij}_{h^{kl}}(h^{kl}, N, \psi, \beta, K^{kl}) \tag{3.4}$$

The $\mathcal{S}_X$ are nonlinear sources. The three first equations, in ψ, $N\psi$ and β^i, stem from the Einstein Hamiltonian and Momentum constraints. In a fully constrained evolution scheme, they are solved at each timestep. The last tensor elliptic equation comes from the Einstein dynamical equation in the case of stationarity. Several works deal with initial data by simply applying the conformal flatness hypothesis, and do not solve it. Here we will solve it numerically for a single black hole numerical spacetime in stationarity. We will use the excision technique for this calculation, and a set of boundary conditions coming from the isolated horizon formalism.

4 Boundary conditions for Einstein Equations, Numerical resolution

We try to simulate the space 3-slice outside of an excised sphere fixed at a radius $r_H = 1$ in our spherical coordinates. Following the prescriptions of Cook and Pfeiffer (2004) and Gourgoulhon & Jaramillo (2006), we find boundary conditions for our elliptic equations (3.2), (3.3),(3.4) by prescribing the sphere to be an isolated horizon slice: If we write $\beta^i = \tilde{b}\tilde{s}^i - V^i$, where $\tilde{s}^i$ is the spacelike outer unit normal to the excised surface, we adapt our coordinates to the geometry of our horizon by setting our time evolution vector to be null on that surface: $\tilde{b} = \frac{N}{\psi^2}$. This ensures the horizon stays at a fixed radius during time evolution. The vanishing of the shear translates into a Dirichlet condition for the other part of the shift, being proportional to the rotational symmetry vector: $V^i = \Omega_r(\frac{\partial}{\partial\phi})^i$, where Ω_r is the rotation rate parameter of the horizon in our coordinates. The vanishing expansion gives an approximate Neumann condition for variable $N\psi$: $4\partial_r(N\psi) = -\frac{N}{\psi}K_{ij}\tilde{s}^i\tilde{s}^j - N\frac{D_i\tilde{s}^i}{\psi^3}$. Finally, we set arbitrarily the boundary condition on the lapse to be a fixed value of 0.3.

There remains the boundary condition on the equation for h^{ij}. We will use for the resolution the scalar variables A and $\tilde{B}$ described in Cordero et al. (2008), and adapted to the Dirac Gauge choice. With the Dirac gauge constraints and an additional determinant condition, the tensor equation reduces to 2 elliptic scalar equations($\mathcal{L}_\beta$ is the Lie derivative along β^i):

$$\Delta A - \frac{\psi^4}{N^2}\mathcal{L}_\beta\mathcal{L}_\beta A = A_{\mathcal{S}}(h^{ij}, N, \psi, \beta, K^{ij}) \tag{4.1}$$

$$\tilde{\Delta}\tilde{B} - \frac{\psi^4}{N^2}\mathcal{L}_\beta\mathcal{L}_\beta \tilde{B} = \tilde{B}_{\mathcal{S}}(h^{ij}, N, \psi, \beta, K^{ij}) \tag{4.2}$$

$\tilde{\Delta}$ is a modified elliptic laplace operator, and $A_{\mathcal{S}}$ and $\tilde{B}_{\mathcal{S}}$ are the A and $\tilde{B}$ potentials associated to the tensor source. Once these two quantities are known, one can entirely reconstruct the tensor h^{ij} using the Dirac gauge and the determinant condition. The gauge is then satisfied by construction at each step.

From these two scalar equations, we have been able to exhibit approximate equations by linear operators acting on A and $\tilde{B}$, and with particular properties. We decompose all our operators and unknowns into spherical harmonics. Using the condition $\tilde{b} = \frac{N}{\psi^2}$, we are able to simplify the double Lie derivative operator and separate radial terms from the others, so that we can approximately write (for example for $A(r,\theta,\phi) = \Sigma A_{lm}(r)Y_l^m(\theta,\phi)$):

$$[-\alpha(r - r_H) - \delta(r - r_H)^2]\frac{d^2}{dr^2}A_{\ell m} + \frac{2}{r}\frac{d}{dr}A_{\ell m} - \frac{\ell(\ell+1)}{r^2}A_{\ell m} = A_{\mathcal{S}} + \frac{\psi^4}{N^2}(\mathcal{L}_\beta\mathcal{L}_\beta A)^{**}_{\ell m}. \tag{4.3}$$

α and δ are real numbers determined by the data.

The ordinary differential operator on the left is singular; a search for analytical homogeneous solutions gives a Kernel of dimension one for usual values of α and δ. By fixing the behaviour of the fields at infinity, there is no further need for a boundary condition on the excised frontier for solving the equation. The same analysis holds for the operator acting on $\tilde{B}$.

This is not an actual proof that our two scalar equations do not need any boundary condition prescription, because the right hand side depends (non-linearly) on the variable. However, we implement our resolution iteratively by inverting at each step these weakly singular operators; our system being convergent, this indicates that indeed, no boundary condition has to be imposed globally for the determination of h^{ij}. The question remains open why it is actually the case, and whether this result applies to more general cases involving isolated horizons.

Our simulation is made on a LORENE 3D spherical grid, using spherical harmonics decomposition for the angular part and spectral multidomain Chebychev decomposition for the radial part. The mapping consists in 4 shells and an outer compactified domain, so that infinity is inside our grid and we have no boundary condition to put at a finite radius. We impose the values of all the fields to be equivalent at infinity to those of a flat 3-space. Except for stationarity, all simulations are done with no assumption of coordinate symmetry.

Having set the shape and the location of the surface in our coordinates, and the lapse being set to a fixed value of 0.3, we are only left with one parameter which is the horizon rotation parameter Ω_r. We generate two sets of initial data, spanning the rotation parameter from zero (Schwarzschild solution) to 0.3. One set will give the solution for the whole differential system, while the other will give the Conformally Flat Data, where instead of solving the equation for the h^{ij} variable, we set it to zero (this is the most commonly used approximation for black hole initial data: however, we know that the Kerr spacetime does not admit any conformally flat slices).

Figure 1 presents the relative accuracy obtained for the Einstein constraints in the non conformally flat case, as well as the accuracy for the Einstein Dynamical equation in the non conformally flat and conformally flat case (this is the only equation not solved in this case). We actually see a major improvement, showing the discrepancy between the usual conformal flatness approximation that one uses generically to simulate rotating spacetimes in numerical relativity, and the actual stationary Kerr solution. Another test of stationarity can be the comparison between the ADM mass and the Komar mass at infinity, the latter being tentatively defined with the supposed Killing vector $(\frac{\partial}{\partial\phi})^i$ (We don't impose any Killing symmetry, except on the horizon: Although we know, by the Black Hole rigidity theorem, that an accurate resolution of Einstein Equations would impose this vector to be so). This is done in figure 2. The comparison between the ADM mass and the Komar mass is

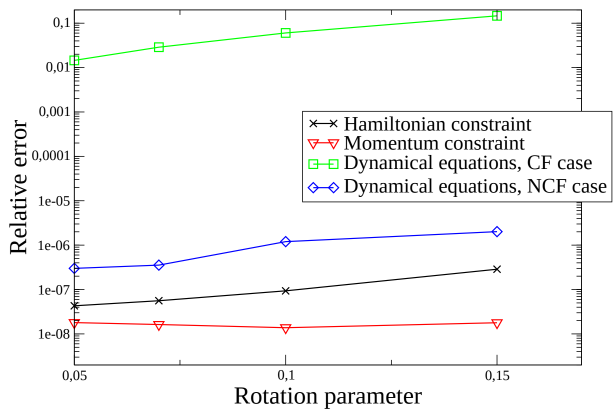

Fig. 1. Accuracy for Einstein Equations resolution

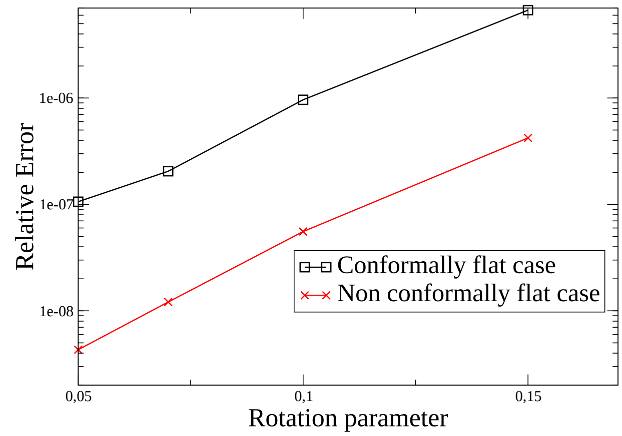

Fig. 2. Relative difference between the ADM and Komar mass in conformally flat and non-conformally flat cases

directly linked to the virial theorem of General Relativity put forth by Gourgoulhon & Bonazzola (1994). The concordance between those masses is equivalent to the vanishing of the virial integral.

We also have computed the accuracy of verification of a Penrose-Like inequality studied in Jaramillo et al. (2007), being the following:

$$\epsilon_A = \frac{\mathcal{A}}{8\pi(M_{ADM}^2 + \sqrt{M_{ADM}^4 - J^2})} \leq 1 \qquad (4.4)$$

$\mathcal{A}$ is the area of the horizon, and M_{ADM} and J are respectively the ADM mass and the Komar angular momentum associated with the 3-slice. Being a little more stringent that the actual Penrose inequality, it is supposed to be verified for all spacetimes containing an apparent horizon, and it is an equality only for actual Kerr apparent horizons. We find an accuracy of 3.10^{-8} at most for this equality in our case. This is another strong hint of the accuracy of our spacetime solution.

To our knowledge, it is the first time the non conformally flat part is numerically computed in a black hole spacetime using only a prescription on the stationarity of the horizon. Further accuracy tests, including geometrical properties of the horizon and the spacetime will be presented in an upcoming paper. The authors warmly thank Eric Gourgoulhon and Jose Luis Jaramillo for numerous fruitful discussions.

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