*SF2A 2011*
*G. Alecian, K. Belkacem, R. Samadi and D. Valls-Gabaud (eds)*

# RELATIVISTIC ASTROMETRY AND TIME TRANSFER FUNCTIONS

S. Bertone[1] and C. Le Poncin-Lafitte[1]

**Abstract.** Given the extreme accuracy reached by future space astrometry missions, one needs a global relativistic modeling of observations. Indeed, a consistent definition of the astrometric observables in the context of General Relativity is essential to find unique stellar coordinates. This is usually done explicitly by solving the null geodesic equations which describe the trajectory of a photon from its emission by a celestial object to its reception by a moving observing satellite. However, we show here that this task can be avoided if one uses the recently developed formalism of the time transfer functions. We describe a possible approach to the reconstruction of the source coordinates from the knowledge of the reception coordinates with this new method.

Keywords: space astrometry, relativity

## 1 Introduction

Future space astrometry missions, such as Gaia (Perryman et al. 2001; Bienayme & Turon 2002) and SIM, will provide large astrometric catalogs with some microarcseconds ($\mu$as) accuracy on positions, parallaxes and proper motions of celestial objects. However it is nowadays well known that $\mu$as astrometry requires a precise relativistic modeling of astrometric parameters. Several of these modelings have been developed during the last decade, such as GREM (Klioner 2003) or RAMOD (de Felice et al. 2006), and they are based on the determination of the light trajectory from the emitting celestial object up to the observing satellite by solving the null geodesic equations. However it has been recently demonstrated that this task is not mandatory and can be replaced with advantages by another approach based on the calculation of the time transfer functions (Le Poncin-Lafitte et al. 2004; Teyssandier & Le Poncin-Lafitte 2008). In this article we illustrate how to build an astrometric modeling with the use of this alternative formalism.

This paper is organized as follows. In section 2 we give the notations used in this article. In section 3 we set up the astrometric problem by introducing a moving observer receiving a light ray from a distant celestial object. Then in section 4, we give the expression of the covariant components of a tangent vector to the light ray received by the observer. In section 5 we deal with the expression of Solar System's gravitational potential, a key point to calculate the light deflection. By introducing a *three zones* modeling, we show how to take into account the motion of planets during the propagation of the light ray from an emitting star to the observing satellite. Finally, we present some concluding remarks in section 6.

## 2 Notations and conventions

Throughout this work, $c$ is the speed of light in vacuum and $G$ is the Newtonian gravitational constant. The Lorentzian metric of space-time $V_4$ is denoted by $g$. We adopt the signature $(+---)$. We suppose that space-time is covered by some global coordinates system $x^\alpha = (x^0, \boldsymbol{x})$, with $x^0 = ct$ and $\boldsymbol{x} = (x^i)$, centered on the Solar System barycenter. Greek indexes run from 0 to 3, and Latin indexes from 1 to 3. Moreover, we assume that the curves of equations $x^i = \text{const}$ are timelike, at least in the neighborhood of the chosen observer. This condition means that $g_{00} > 0$ in the vicinity of the observer. Any ordered triple is denoted by a bold letter. In order to distinguish the triples built with the spacelike contravariant components of a vector from the ones built with covariant components, we systematically use the notation $\boldsymbol{a} = (a^1, a^2, a^3) = (a^i)$ and $\underline{\boldsymbol{b}} = (b_1, b_2, b_3) = (b_i)$.

[1] Observatoire de Paris - SYRTE, CNRS/UMR8630 , 77 Av. de l'Observatoire, 75014 Paris, France

## 3 Set up of the astrometric problem

Let us consider a timelike observer. Along its wordline we introduce a comoving tetrad of four vectors $E^{\alpha}_{\hat{\mu}}$ where index $(\mu)$ is only the tetrad index running from 0 to 3, to enumerate the four 4-vectors, while index $\alpha$ is a normal tensor index. We postulate that $E^{\alpha}_{(0)}$ is a timelike vector strictly equal to the unit four-velocity $u^{\alpha}$ of the satellite. When the observer receives a light ray from a distant object, one can characterize its spacetime direction by a null vector $k^{\mu} = (k^0, k^i)$ tangent to the incoming light ray. Its unit spacelike direction $\tilde{k}^{\mu}$ relative to the hyperplane orthogonal to $u^{\alpha}$ is then given by (Teyssandier & Le Poncin-Lafitte 2006)

$$\tilde{k}^{\mu} = \frac{k^{\mu}}{u^{\nu}k_{\nu}} - u^{\mu}\,. \tag{3.1}$$

It is now straightforward to calculate the three director cosines $\cos\phi_{(a)}$ formed by each spacelike vector $E^{\alpha}_{(a)}$ (here index $(a)$ runs from 1 to 3) with $\tilde{k}^{\mu}$, as follows

$$\cos\phi_{(a)} = -\frac{E^0_{(a)} + \hat{k}_i E^i_{(a)}}{u^0\left(1 + \hat{k}_i\beta^i\right)}\,, \tag{3.2}$$

where $\beta^i = v^i/c$, $v^i$ being the coordinate velocity of the observer, and $\hat{k}_i = k_i/k_0$ will be called in the following the deflection functions. We immediately deduce from equation (3.2) that the knowledge of the ratio $k_i/k_0$ fully characterizes the light ray at the reception point and so it is mandatory to determine completely the astrometric director cosines. However, most of relativistic modelings are dealing with an explicit integration of the light ray equations to calculate this ratio, even if these calculations may be very complicated. Thus, our purpose is now summarized by this question: *if we do not determine the light ray trajectory from its emission to its reception, what can we do instead?* The answer lies in the basic properties of a null geodesic connecting two point-events, in particular the relationship between time delay and deflection of light.

## 4 The covariant components of the tangent vector

Let us consider that spacetime is globally regular with the topology $\mathbb{R}\times\mathbb{R}^3$ and that it is without horizon, which is admissible in the context of practical space astrometry within the Solar System. Henceforth, we are working with a barycentric coordinates system (BCRS). In this section, we consider only one deflecting body of spatial position $\boldsymbol{x}_p$ and we suppose the existence of a unique light ray connecting two point-events $x_A = (ct_A, \boldsymbol{x}_A)$ and $x_B = (ct_B, \boldsymbol{x}_B)$. By convention $x_A$ and $x_B$ denote the emission point and the reception point of the photon, respectively. In addition we put $R_{AB} \equiv R = |\boldsymbol{x}_B - \boldsymbol{x}_A|$ as illustrated in figure 1.

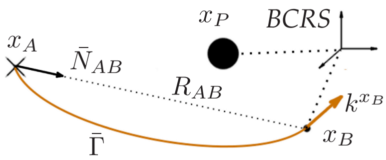


**Fig. 1.** Illustration of a light deflection experiment.

Working at the post-Newtonian approximation of General Relativity and considering the metric tensor recommended by the International Astronomical Union (Soffel et al. 2003), we obtained an expression of the deflection functions $\hat{k}_i$ at reception point by calculating the derivatives of the time transfer functions as follows (Bertone & Le Poncin-Lafitte 2011)

$$\begin{aligned}\hat{k}_i = &-N^i \\ &-\frac{(\gamma+1)}{c^2}\int_0^1\left[WN^i + (1-\lambda)R\frac{\partial W}{\partial x^i}\right]_{z^{\alpha}_{-}(\lambda)} d\lambda \\ &+\frac{4}{c^3}\int_0^1\left[W^i - \frac{(\gamma+1)}{4}(1-\lambda)RN^i\frac{\partial W}{\partial t}\right. \\ &\left.+(1-\lambda)R\left(\boldsymbol{N}\,.\frac{\partial \boldsymbol{W}}{\partial x^i}\right)\right]_{z^{\alpha}_{-}(\lambda)} d\lambda + \mathcal{O}\left(c^{-4}\right)\,,\end{aligned} \tag{4.1}$$

where $\gamma$ is a PPN parameter (Will 1993) and the integral is taken along the Minkowskian line of sight

$$z_{-}(\lambda) = (ct_B - \lambda R, \boldsymbol{x}_B - \lambda R\boldsymbol{N}) \,, \tag{4.2}$$

with $\boldsymbol{N} = \{N^i\} = (\boldsymbol{x}_B - \boldsymbol{x}_A)/R$, $W$ and $\boldsymbol{W}$ being the gravitational potentials studied in the next section.

## 5 Towards a "three zones" modeling

Here we assume the Solar System as an isolated system of $N$ deflecting bodies, thus we neglect gravitational perturbations due to the galaxy, dark matter, the emitting star itself, etc. Then, potentials $W$ and $\boldsymbol{W}$ are generated only by the Solar System bodies and can be written

$$W(t,\boldsymbol{x}) = \sum_{p=0}^{N} \frac{GM_p}{r_p}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_e^p}{r_p}\right)^n P_n\left(\frac{\mathbf{k}_p\cdot\boldsymbol{x}}{r_p}\right)\right], \tag{5.1}$$

$$\boldsymbol{W}(t,\boldsymbol{x}) = \sum_{p=0}^{N} \frac{GM_p\boldsymbol{v}_p}{r_p}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_e^p}{r_p}\right)^n P_n\left(\frac{\mathbf{k}_p\cdot\boldsymbol{x}}{r_p}\right)\right], \tag{5.2}$$

where $P_n$ are the Legendre polynomials, $r_p = |\boldsymbol{x} - \boldsymbol{x}_p(t)|$ is the spatial distance between the light signal and the perturbing body $p$ at time $t$; $\mathbf{k}_p$, $M_p$, $J_n^p$ and $r_e^p$ denote the unit vector along the axis of symmetry, the mass, the mass multipole moments and the equatorial radius of body $p$, respectively.

To calculate the total deflection of light, we need to know the motion of Solar System bodies from the emission to the reception of the light ray. But if the light is coming from a star, it is quite impossible to give a simple analytical expression of that motion and we have to use an ephemeris. That is why we propose a *three zones* modeling, as illustrated on figure 2, to deal with this problem:

- **internal zone**: let us introduce a fictitious point-event $x_C = (ct_C, \boldsymbol{x}_C)$ on the light world line. Spatial distance of $x_C$ from Solar System barycenter is supposed to be small, such as 100 astronomical units. The deflection functions $\hat{k}_i$ at $x_B$ is known since the satellite has observed a celestial object. Then, one uses equation (4.1) to compute $x_C$ when we consider the potentials given by equations (5.1) and (5.2). $t_C - t_B$ is a small time interval, so we can approximate the motion of Solar System bodies by a straight line of equation
$$\boldsymbol{x}_p(t) = \boldsymbol{x}_p^B - (t - t_B)\boldsymbol{v}_p^B \,, \tag{5.3}$$
where $\boldsymbol{x}_p^B$ and $\boldsymbol{v}_p^B$ are the position and the velocity of body $p$ at time $t_B$, respectively. In this case, we can derive an analytical formula for equation (4.1).

- **external zone**: we introduce another fictitious point-event $x'_C = (ct'_C, \boldsymbol{x}'_C)$ on the light world line. Here $x'_C$ is assumed to be very far from the Solar System in such a way that its gravitational field can be approximated by a Schwarzchild's metric. This very well known problem has an analytical solution which can be obtained easily with the time transfer functions (Le Poncin-Lafitte & Teyssandier 2008).

- **third zone**: this zone is introduced to enable a smooth transition between the internal and external zones. To achieve this, we want to find a criterion to match the deflection functions at $x_C$ and $x_{C'}$, respectively. A possible choice is to stipulate that the angular distance between them must be one tenth below the desired astrometric accuracy.

## 6 Conclusion

In this article, we have presented a relativistic modeling for high precision space astrometry, valid in the Post-Newtonian approximation. This approach allows us to take advantage of a simple motion law for the bodies during the light propagation through the Solar System, i.e. the internal zone. We also outline a matching procedure between this zone and an additional external area where Solar System bodies are supposed at rest. In this work, light deflection is obtained as a boundary value problem solved by the use of the time transfer functions and consequently a complete solution of the light ray trajectory is avoided.

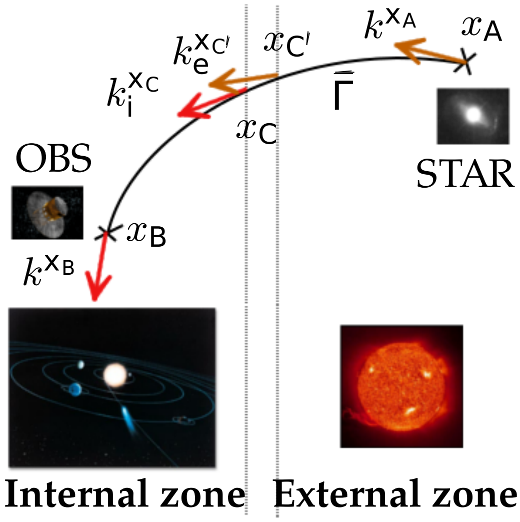


**Fig. 2.** The *three zones* modeling.

## References

Bertone, S. & Le Poncin-Lafitte, C. 2011, Memorie della Societa' Astronomica Italiana, to be submitted

Bienayme, O. & Turon, C. 2002, EAS Publications Series, 2

de Felice, F., Vecchiato, A., Crosta, M. T., Bucciarelli, B., & Lattanzi, M. G. 2006, Astrophysical Journal, 653, 1552

Klioner, S. A. 2003, Astronomical Journal, 125, 1580

Le Poncin-Lafitte, C., Linet, B., & Teyssandier, P. 2004, Classical and Quantum Gravity, 21, 4463

Le Poncin-Lafitte, C. & Teyssandier, P. 2008, Physical Review D, 77, 044029

Perryman, M. A. C., de Boer, K. S., Gilmore, G., et al. 2001, Astronomy and Astrophysics, 369, 339

Soffel, M., Klioner, S. A., Petit, G., et al. 2003, Astronomical Journal, 126, 2687

Teyssandier, P. & Le Poncin-Lafitte, C. 2006, ArXiv General Relativity and Quantum Cosmology e-prints

Teyssandier, P. & Le Poncin-Lafitte, C. 2008, Classical and Quantum Gravity, 25, 145020

Will, C. M. 1993, Theory and Experiment in Gravitational Physics, ed. C. M. Will