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POTENTIAL GENERATED INNER AND OUTSIDE A CIRCULAR WIRE IN ITS PLANE. APPLICATION TO SATURN'S RING

N.-E. Najid[1], M. Zegoumou[1] and E.H. El Ourabi[1]

Abstract. In this article we derive the development of the potential generated by a homogeneous wire bent into a circular shape (Najid, Jammari & Zegoumou, 2005). We develop the potential as a power series of the distance from an appropriate origin to the test particle. The potential is expressed as a function of Legendre polynomials. We study both, the case where the test particle is inside or outside the circular wire. By Lagrangian formulation, we establish the differential equation of motion. The numerical resolution leads us to different orbits. Outside the wire we get a case where the test particle is confined between a maxima and minima of the radial position; while inner the wire the test particle is subjected to an escape case depending on the time of integration.

Keywords: Potential, Legendre Polynomial, Orbits.

1 Introduction

The irregular shapes of many celestial bodies (Kellog 1954)have gained a great interest during the last decades. Their physical and geometrical studies require an accurate knowledge of the potential generated by them (Danby 1992). In our study, we develop the method of calculation of the potential generated by a circular wire in a point located at the plane of the wire. The result is given directly by a series expansion in terms of R, the radius of the wire and his total mass. We are interested to the points outside and inner the circle. We established the equation of motion of a test particle and give the orbits in accordance with the initial conditions. Precession of perihelia or chaotic cases is proved (Murray & Dermott 1999).

2 Potential generated by a circular wire

We consider a circular ring of radius R and total mass M, located in the (xoy) plane (Fig.1). The gravitational potential generated by the ring at a point M (x, y) is expressed by:

$$dU = -G\frac{dm}{PM} \tag{2.1}$$

ρ The distance between the element dm centered at P, (Fig.1) and M.

- *Expression of* ρ :

$PM^2 = OM^2 + OP^2 - 2OM.OP\cos(\overrightarrow{OM}, \overrightarrow{OP})$

$$\rho = PM = \sqrt{r^2 + R^2 - 2.r.R.\cos(\theta - \psi)} \tag{2.2}$$

[1] Laboratoire de Physique Théorique et Appliquée Université Hassan II Ain Chock- Faculté des Sciences Ain Chock BP 5366 Maârif Casablanca Morocco

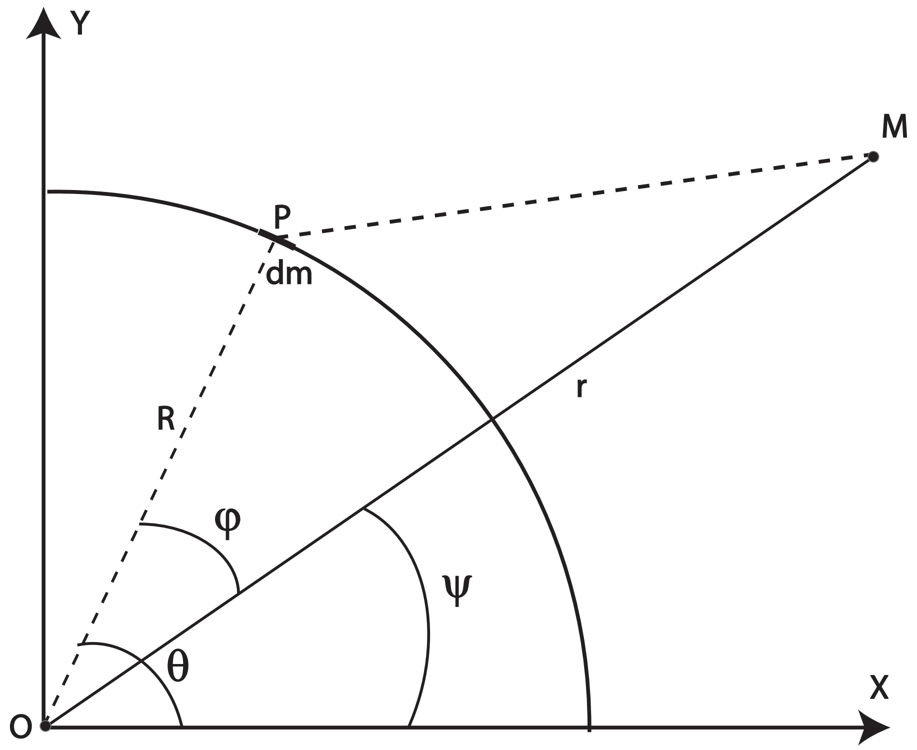

Fig. 1. Ring in the plane (xoy)

- Expression of dm:

We have:

$$dm = \lambda dl = \lambda R d\theta \tag{2.3}$$

Substituting the expressions (2.2) and (2.3) in (2.1), the potential generated by the ring is:

$$U = -G.R.\lambda \int_0^{2\pi} \frac{d\theta}{\sqrt{r^2 + R^2 - 2.r.R.\cos(\theta - \psi)}} \tag{2.4}$$

3 The potential created by a circular ring at a point outside of the ring

By means of Legendre polynomials(MacRobert 1927) in (2.4), we get:

$$\frac{1}{\sqrt{r^2 + R^2 - 2.r.R.\cos(\varphi)}} = P_0(\cos\varphi).\frac{1}{r} + P_1(\cos\varphi).\frac{R}{r^2} + P_2(\cos\varphi).\frac{R^2}{r^3} + ...$$

Limiting ourselves to order 2 we can write:

$$U = -G\lambda R \left[\int_{-\psi}^{2\pi-\psi} \frac{d\varphi}{r} + \int_{-\psi}^{2\pi-\psi} \frac{R}{r^2} \cos\varphi d\varphi + \int_{-\psi}^{2\pi-\psi} \frac{R^2 \left(\cos^2\varphi - 1\right)}{2r^3} d\varphi \right]$$

After integration we find:

$$U = -\frac{G.\lambda.R}{r}.2\pi - \frac{G.\lambda.R^3}{4.r^3}.2\pi + ... \tag{3.1}$$

We add to the potential created by the ring the potential created by the planet.
This is a Keplerian potential: $U_{planet} = -\frac{G.M_{planet}}{r}$

With $M = \int dm = \int_0^{2\pi} \lambda R d\theta = 2.\pi.R.\lambda$, we arrive to :

$$U = -\frac{GM}{r} - \frac{GMR^2}{4r^3} - \frac{G.M_{planet}}{r} = -\frac{G\left(M + M_{planet}\right)}{r} - \frac{GMR^2}{4r^3}$$

$$U = \frac{A}{r} + \frac{B}{r^3} \tag{3.2}$$

The expression (3.2) represents the form of the potential, with:

$A = -G\left(M + M_{planet}\right)$ and $B = -\frac{GMR^2}{4}$.
U is viewed as two parts, one consist of the keplerian case, while the other summaries the perturbation.

3.1 The Lagrangian of the particle test

We study the dynamical behavior of a test particle, with unit mass, in the field of the homogeneous ring. The Lagrangian of the test particle is given by:

$L = T - U$, so:

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\psi}^2 - \frac{A}{r} - \frac{B}{r^3} \tag{3.3}$$

3.2 Equation of motion

The equations of motion are given by: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$
with: $\frac{\partial L}{\partial r} = r\dot{\psi}^2 - \frac{G(M+M_{planet})}{r^2} - \frac{3GMR^2}{4r^4}$
and $\frac{\partial L}{\partial \dot{r}} = \dot{r}$
Therefore: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \ddot{r}$ and subsequently we find:

$$\ddot{r} = r\dot{\psi}^2 - \frac{G\left(M + M_{planet}\right)}{r^2} - \frac{3GMR^2}{4r^4} \tag{3.4}$$

With $u = \frac{1}{r}$, we have $\ddot{r} = h^2u^2\frac{d^2u}{d\psi^2}$ and $\dot{\psi} = h.u^2$
After calculation and arrangement of expressions (3.4) we find, the differential equation of motion

$$\frac{d^2u}{d\psi^2} + \frac{3B}{h^2}u^2 + u = -\frac{A}{h^2} \tag{3.5}$$

Finally we can write this equation as:

$$\frac{d^2u}{d\psi^2} + au^2 + u = b \tag{3.6}$$

Where $a = \frac{3B}{h^2} < 0$ will be examined as a term of a small perturbation.
And $b = -\frac{A}{h^2} > 0$ corresponds to the well-known keplerian case.
The equation (3.6) represents the dynamical equation of motion of the test particle in the gravitational field generated by the homogeneous ring. This equation is nonlinear, she require then a numerical resolution.

3.3 Trajectories

Keplerian case: a=0

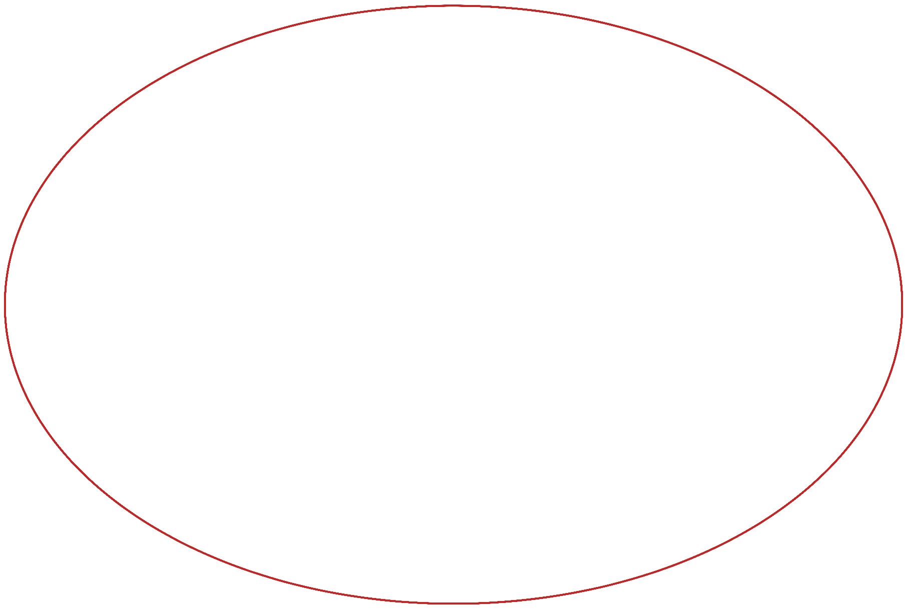

Fig. 2.

The figure 2, correspond to an elliptic orbit. This is a regular periodic orbit. The corresponding potential is $\frac{\mathrm{A}}{\mathrm{r}}$

General case:

From different initial conditions, we reach many different cases as in figures 3, 4, 5 and 6.

Figure 3 correspond to a precession of the perihelia.

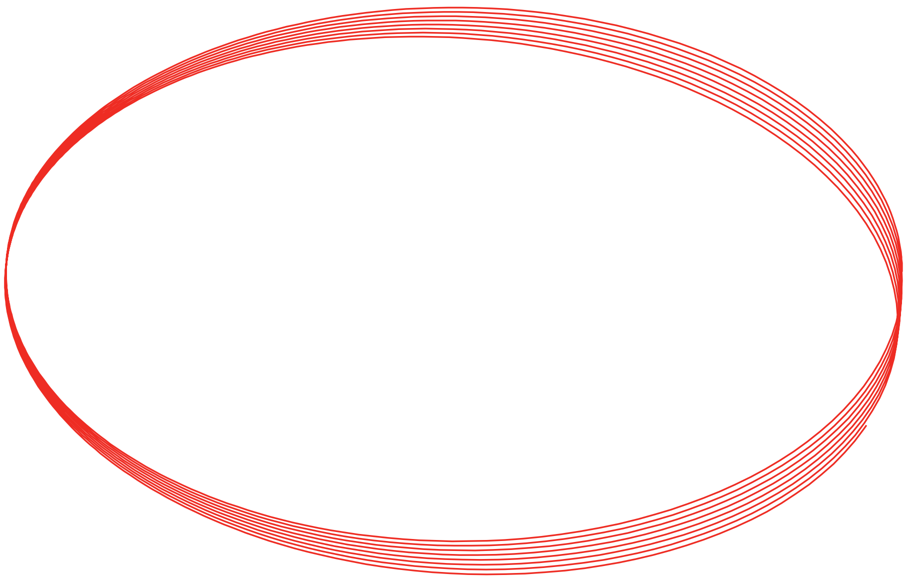

Fig. 3.

We find in fig (4, 5, 6) that when the term "a" increases the test particle is confined between two elliptic boundary trajectories. However, we approach a limit trajectories, when the disturbance term increases.

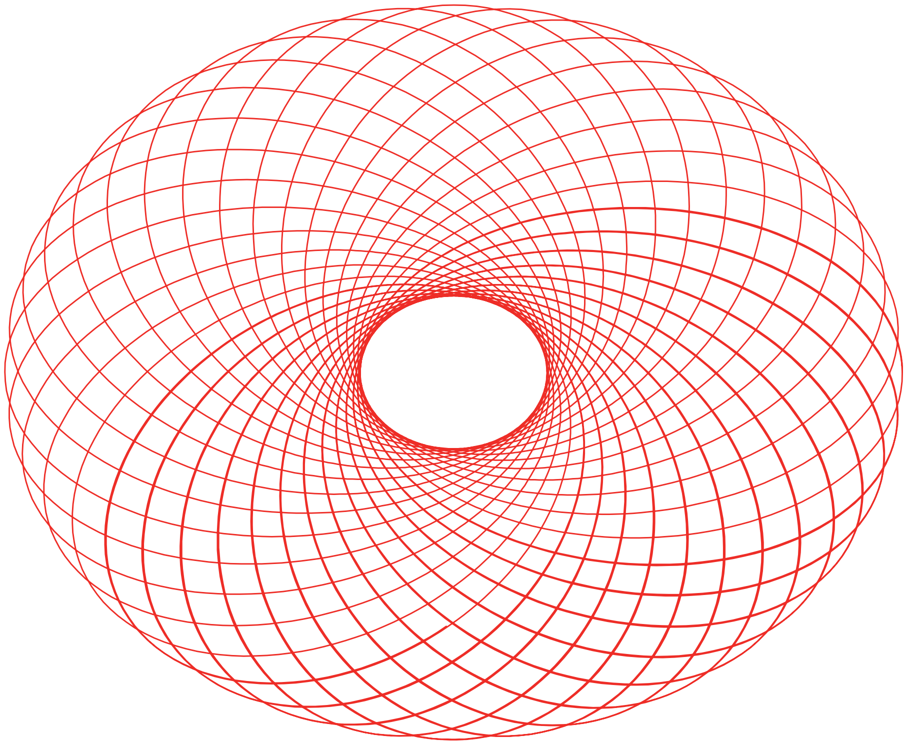

Fig. 4.

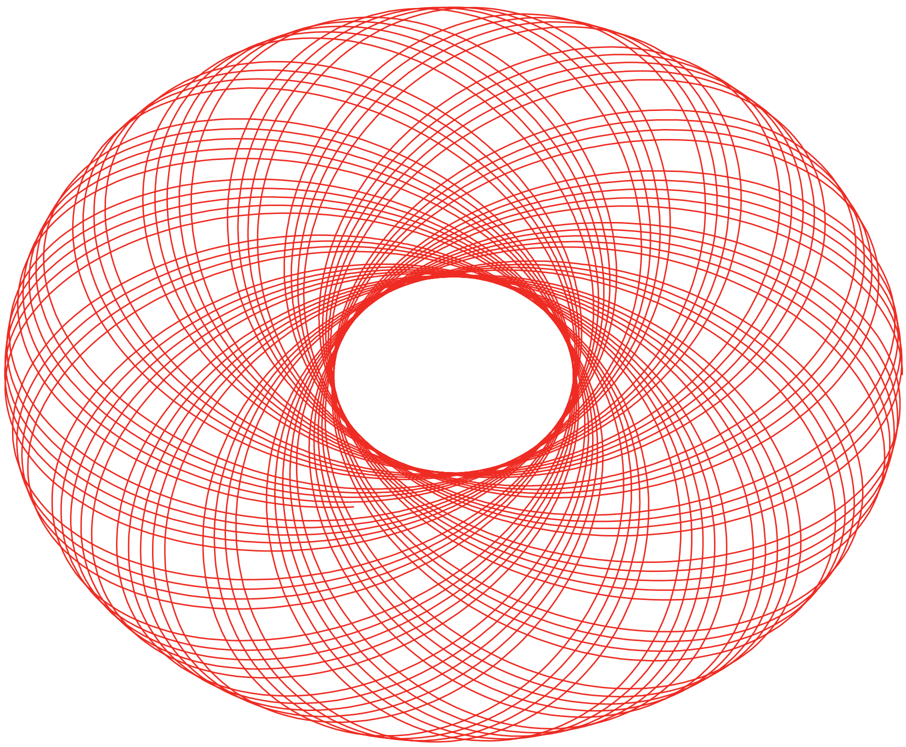

Fig. 5.

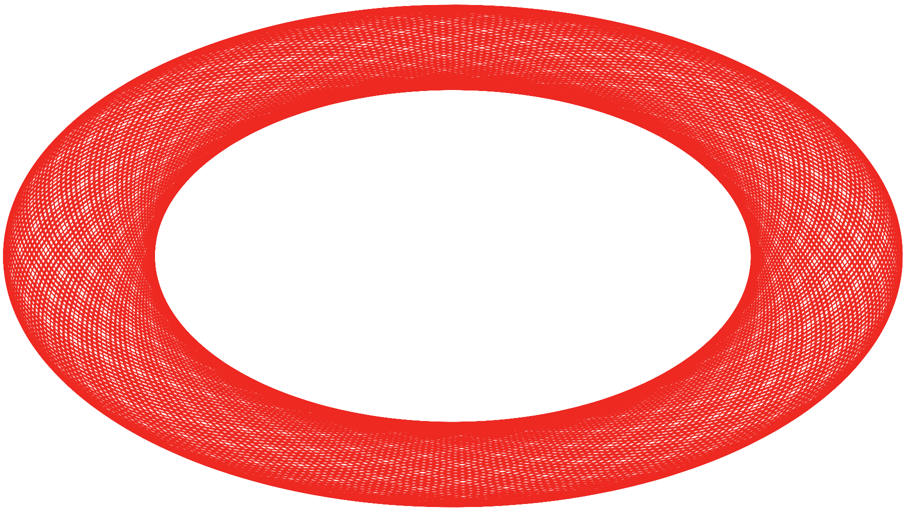

Fig. 6.

4 The potential created by a circular ring at a point inner the ring

4.1 The potential expression

The expression $\frac{1}{\sqrt{r^2+R^2-2.r.R.\cos(\varphi)}}$ is written again in this case as:

$$\frac{1}{\sqrt{r^2+R^2-2.r.R.\cos(\varphi)}} = P_0(\cos\varphi).\frac{1}{R} + P_1(\cos\varphi).\frac{r}{R^2} + P_2(\cos\varphi).\frac{r^2}{R^3} + P_3(\cos\varphi).\frac{r^3}{R^4} + ...$$

With $P_n(\cos\varphi)$ are the coefficients of LEGENDRE.
The expression of the potential becomes:

$$U = -G\lambda R \left[\begin{array}{l} \int_{-\psi}^{2\pi-\psi} \frac{d\varphi}{R} + \int_{-\psi}^{2\pi-\psi} \frac{r\cos\varphi}{R^2} d\varphi \\ + \int_{-\psi}^{2\pi-\psi} \frac{r^2\left(3\cos^2\varphi-1\right)}{R^3} d\varphi + \int_{-\psi}^{2\pi-\psi} \frac{r^3\left(5\cos^3\varphi-3\cos\varphi\right)}{R^4} d\varphi \end{array} \right]$$

After calculation we arrive at the following expression

$$U = -\frac{GM}{R} - \frac{GMr^2}{4R^3} = -\frac{GM}{R^3}\left(R^2 + \frac{r^2}{4}\right) \tag{4.1}$$

By adding the potential generated by the ring to that of the central planet, we get:

$$U = -\frac{G.M}{R^3}\left(R^2 + \frac{r^2}{4}\right) - \frac{G.M_{planet}}{r}$$

4.2 Equation of motion

As before, we arrive for a test particle of unit mass to the expression:

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\psi}^2 + \frac{G.M}{R^3}\left(R^2 + \frac{r^2}{4}\right) + \frac{GM_{planet}}{r}$$

The equations of motion are given by:

$$\frac{\partial L}{\partial r} = r\,\dot{\psi}^2 + \frac{G.Mr}{2R^3} - \frac{GM_{planet}}{r^2}$$

We have $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$ with $\frac{\partial L}{\partial r} = r\,\dot{\psi}^2 + \frac{G.Mr}{2R^3} - \frac{GM_{planet}}{r^2}$
and $\frac{\partial L}{\partial \dot{r}} = \dot{r}$ .Therefore: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \ddot{r}$ and subsequently we find:

$$\ddot{r} = r\,\dot{\psi}^2 - \frac{GM_{planet}}{r^2} + \frac{G.Mr}{2R^3} \tag{4.2}$$

By making the change of variable $u = \frac{1}{r}$,we can write the differential equation as follows:

$$\frac{d^2u}{d\psi^2} + u + \frac{GM}{2R^3h^2u^3} - \frac{GM_{planet}}{h^2} = 0$$

$$\frac{d^2u}{d\psi^2} + u + \frac{a}{u^3} + b = 0 \tag{4.3}$$

With : $a = \frac{G.M}{2.h^2.R^3} > 0 \quad ; \quad b = -\frac{GM_{planet}}{h^2} < 0$

5 Conclusions and perspectives

- In this work we studied the dynamical behavior of a test particle in the gravitational potential of a homogeneous circular wire. We got a few orbits depending on initial conditions.
- If r> R it was found that the test particle is confined in space between two limit trajectories, with no possibility of escape.
- Similarly for r < R we found that for specific initial conditions we obtain a closed trajectory after some laps, while for other initial conditions the test particle is located in space without a limit cycle.
- To analyze these orbits we have to use quantitative tools such as the method of Poincaré section.
- For a more realistic model, we must consider a three-dimensional wire (Pascoli 2000).
- Also, to study the behavior near the rings of Saturn, a study of a disk is in progress.

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