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ANALYTICAL EXPRESSIONS AND NUMERICAL SIMULATIONS FOR AN EXTERNAL CIRCULAR OCCULTER CORONAGRAPH

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Abstract. We give in this study an analytical expression for the Fresnel diffraction of a circular occulter and compare it to numerical computations. The analytical expression uses Lommel series. Two different series are needed, whether the inside or the outside of the occulter's geometric shadow is considered. These series are infinite sums of Bessel functions which converge more or less rapidly depending on the experimental parameters and the position in the shadow. Numerical computations require thousands of points. The calculations given here are valid for an incoming plane wave, i.e. a point-like source. This is the necessary first step to a generalization to solar studies. The good consistency observed between numerical computations and analytical calculations makes it possible to generalize the numerical computations to different forms of occulters, and an example is given for a serrated edge disk. Simple considerations suggest that shaped apertures are required for exoplanet detection while mere circular occulters are convenient for solar coronal studies.

Keywords: Sun, exoplanets, coronagraph, external occulter

1 Introduction

The problem of observing the solar corona has been addressed by solar astronomers since the fundamental experiment of Lyot (1939) and the first use of external occulter by Evans (1948). An extremely well documented description of the methods of observation of these techniques, and the history of solar coronagraphy can be found in Koutchmy (1988). Some of the ideas described in this review paper (for example shaped occulters) are now considered for the detection of exoplanets (see for example Arenberg et al. (2007)). Shaped occulters have been proposed independently for solar and extrasolar studies the same year (Spitzer 1962; Purcell & Koomen 1962).

The problems are technically much more difficult for the exoplanet case than for the solar case, but from a computational point of view, the difficulties are inverted. For the exoplanet study, the star is considered as an unresolved point-like source. If the objective is to see an Earth-like planet at 10 parsecs, then the occulter as seen from the telescope must create a shadow with an angle of 0.2". From this constraint, and for a 4-m telescope, external occulters must have a 10 to 50 m diameter and be kept at a distance of 10 000 to 80 000 km.

For solar observations, the angle of the shadow is that of the entire solar disk, 1/100 rd, about 10 000 times larger than for an exoplanet. The occulter diameter becomes 1 to 1.5 m., and it must be kept at 100 to 150 m from a small telescope, as described in Lamy et al. (2010). These conditions correspond to a large number of Fresnel zones on the occulter, and the diffraction is much more difficult to compute. Another major difficulty is the fact that the Sun is an extended source and that it is necessary to sum contributions coming from all parts of the solar disk to obtain the resulting intensity at the position of the corona.

We consider here the case of a point-like source, a preliminary step that is required to compute the actual performance of a solar occulter. Nevertheless, interesting inferences can be already obtained from this first work.

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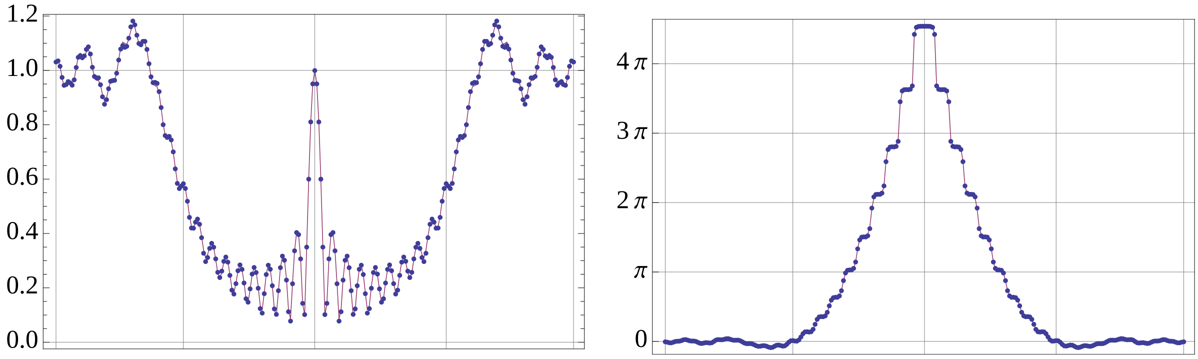

Fig. 1. Modulus (**left**) and unwrapped phase (**right**) produced by a circular occulter of diameter 10 m set at 10 000 km for an incident wave plane. The wavelength is $0.55\mu m$. The Arago spot is dominant in this configuration. The two vertical lines gives the geometrical position of the edges of the occulter. Continuous curves correspond to Eq. 2.3 while dots are the result of a numerical simulation making use of the spatial filtering.

2 Analytic expressions for the Fresnel diffraction of a circular occulter.

The shadow produced at the telescope entrance aperture by an external occulter is a typical problem of Fresnel diffraction, i.e. the free-space propagation of a coherent wave between two planes over a finite distance. Let us briefly recall some fundamentals equations.

If we denote $f_0(x, y)$ the complex amplitude of a wave in the plane $z = 0$ (the plane of the occulter in our problem), its expression $f_z(x, y)$ at the distance z (the entrance aperture of the telescope) can be obtained by any of the equivalent equations:

$$\begin{aligned} f_z(x,y) &= \frac{1}{i\lambda z}\iint f_0(\xi,\eta)\exp\frac{i\pi(\xi^2+\eta^2)}{\lambda z}\exp(-2i\pi\frac{\xi x+\eta y}{\lambda z})\,d\xi\,d\eta \\ &= f_0(x,y) * \frac{1}{i\lambda z}\exp\frac{i\pi(x^2+y^2)}{\lambda z} \\ f_z(x,y) &= \Im^{-1}\left[\hat{f}_0(u,v)\exp(-i\pi\lambda z(u^2+v^2))\right] \end{aligned} \tag{2.1}$$

where λ is the wavelength of the light, the symbol $*$ stands for the 2D convolution, $\hat{f}_0(u, v)$ is the 2D Fourier transform of $f_0(x, y)$ for the spatial frequencies (u, v), and $\Im^{-1}$ denotes the operation of inverse 2D Fourier transform.

The first relation is the mathematical formulation of the Huygens-Fresnel principle, which can be written as a convolution relationship. The integral will be used for the derivation of the analytical expression of the Fresnel diffraction of a circular occulter. The inverse filtering formula is suitable for numerical computation, because it allows the use of fast Fourier transforms. A numerical difficulty arises because of the very high sampling required either by the quadratic phase term filter $\exp(-i\pi\lambda z(u^2 + v^2))$ or by $\hat{f}_0(u, v)$.

The transmission of a circular occulter of diameter D can be written as $1 - \Pi(r/D)$, where $r = \sqrt{x^2 + y^2}$, and $\Pi(r)$ is the top-hat function of transmission 1 for $|r| < 1/2$, and 0 elsewhere. Making use of the first expression of Eq. 2.1, the amplitude $\Psi(r)$ of the wave produced by the Fresnel diffraction of the occulter at the distance z can be written as:

$$\Psi(r) = A - \frac{A}{i\lambda z}\exp(i\pi\frac{r^2}{\lambda z})\int_0^{D/2} 2\pi\xi\exp(i\pi\frac{\xi^2}{\lambda z})J_0(2\pi\frac{\xi r}{\lambda z})d\xi \tag{2.2}$$

where $J_0(r)$ is the Bessel function of the first kind and A stands for $A\exp(2i\pi z/\lambda)$. As we can see, for a wave of unit amplitude ($A = 1$), the Fresnel diffraction of the occulter writes as 1 minus the Fresnel diffraction of the hole. In the literature, it is often inadequately made reference to the Babinet theorem for this property.

Obtaining the complete expression of the wave for any r value is not straightforward. It is easy to show that, at the center of the shadow ($r = 0$), there is a bright spot, of the same intensity level as if the occulter was not present, and which is the famous Arago spot, experimental corner stone of Fresnel's theory. The integral in Eq. 2.2 is a Hankel transform that does not have a simple analytical solution for $r \neq 0$.

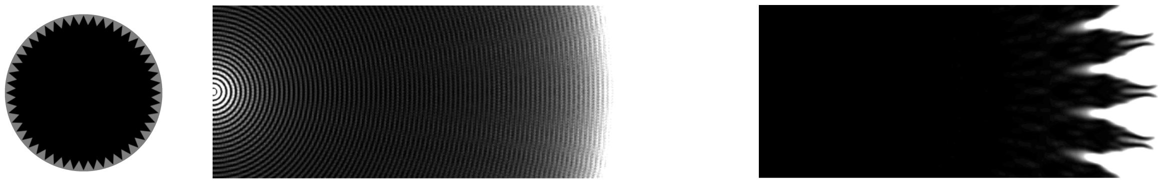

Fig. 2. Left: the circular occulter and the serrated edge outlined inside. Center: the shadow produced by the circular occulter. **Right:** the shadow produced by the serrated edge occulter. The figure is for illustration only, the parameters (occulter of 10 cm at 100 m) being chosen for computational constraints and easiness of representation.

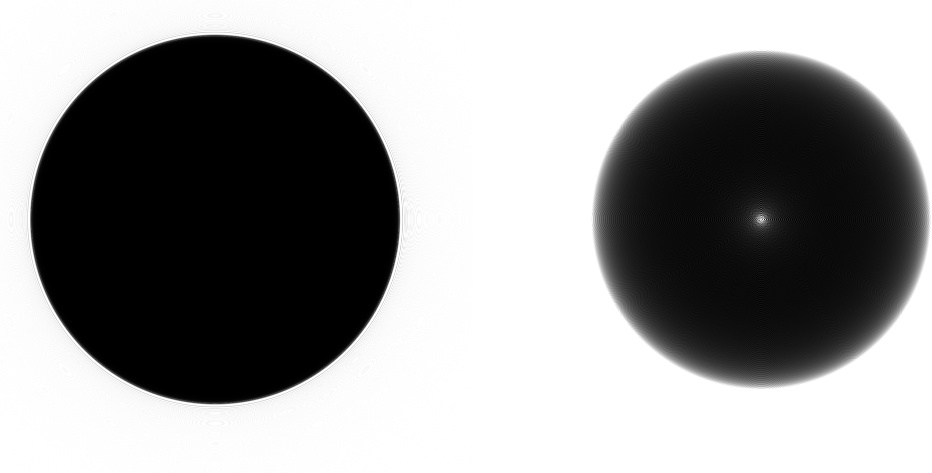

Fig. 3. Shadow produced by a circular occulter of diameter 1 m set at 100 m for an incident wave plane. **Left:** direct representation. **Right:** Same as left panel with the intensity multiplied by 1000.

A similar problem is found in the three-dimensional light distribution near focus, described in Born & Wolf (2006). It makes use of Lommel series. We have used this approach to obtain the Fresnel diffraction of a circular occulter. After some calculations, we obtain the simplified expressions for a circular occulter of diameter D at the distance z:

$$\begin{aligned}
&\text{for } r < D/2: \quad \Psi(r) = A\exp(i\frac{\pi r^2}{\lambda z})\exp(i\frac{\pi D^2}{4\lambda z}) \times \sum_{k=0}^{\infty}(-i)^k(\frac{2r}{D})^k J_k(\frac{\pi D r}{\lambda z}) \\
&\text{for } r = D/2: \quad \Psi(\frac{D}{2}) = \frac{A}{2}(1+\exp(i\frac{\pi D^2}{2\lambda z})J_0(\frac{\pi D^2}{2\lambda z})) \\
&\text{for } r > D/2: \quad \Psi(r) = A - A\exp(i\frac{\pi r^2}{\lambda z})\exp(i\frac{\pi D^2}{4\lambda z}) \times \sum_{k=1}^{\infty}(-i)^k(\frac{D}{2r})^k J_k(\frac{\pi D r}{\lambda z})
\end{aligned} \tag{2.3}$$

To the authors' knowledge, these expressions have not yet been published in the literature.

The two series, for the inner ($r < D/2$) and outer ($r > D/2$) zones of diffraction, converge to the correct value for $r = D/2$. However, in practice, the series are limited to a finite number of terms and the convergence of the series is more difficult to obtain near the transition zone than far away from it. In practice, we used up to 400 terms for the series, which corresponds to a reasonable computing time and a precise result. All computations were made using *Mathematica*[i].

In Fig. 1 we have represented the modulus and the (unwrapped) phase of the shadow produced by an incoming plane wave occulted by a 10m. circular occulter located at 10 000 km from the telescope, parameters

[i] *Mathematica* 2012, Wolfram Research, Inc., Champaign, IL

corresponding to the observation of an exoplanet. The continuous lines are obtained from the Lommel series (100 terms) and the dots are from a numerical computation. The behavior of the phase in the Fresnel diffraction of the occulter is particular. Although the phase term is almost zero in the bright part outside the disk, it becomes highly perturbed in the shadow behind the screen. For the detection of an exoplanet, this means that the telescope response will be quite different for the star and the planet, and the question of a wavefront correction in the form of a multi-object adaptive optics system may be raised.

As it can be seen in the figure, the shadow is not dark enough to suppress the starlight, and the Arago spot is very bright. This is the reason why much larger occulters (for example 50 m. at 80 000 km) are considered. Moreover, occulters with a shaped contour will be preferred to circular ones for exoplanet detection. They have the advantage of producing a much darker central region. An illustration of that is given in Fig. 2 for a serrated edge occulter, similar to the one described in Koutchmy (1988).

Such occulters, that are excellent for the detection of an exoplanet, are probably inappropriate for the Sun that is an extended source. Each point of the solar surface will produce a shifted shadow on the telescope aperture. The goal is to cancel the light coming from the solar disk without attenuating the light coming from the corona. Creating a dark shadow at the center of the figure is not as crucial as having a sharp transition between the dark and bright areas. As it is clearly visible in the figure, a serrated edge occulter does not induce a sharp transition between the dark and the bright regions. However, this was not the conclusion of Fort et al. (1978) considering an occulter with very small teeth, and further studies of these kind of occulters are needed.

The use of a circular occulter in the case of solar observations is nevertheless a possible solution. In Fig. 3 (left) we have represented the intensity produced by an incoming plane wave on a circular occulter with a diameter that ranges from 1 m to 100 m, parameters corresponding to the shadow produced by a point source of the solar disk. The Arago spot is no more visible in this figure, but remains visible in Fig. 3 (right) where the intensity level has been multiplied by a thousand. The geometry of the two figures are the same, which means that the edges of the dark zone, that appeared sharp in the left figure are indeed somewhat smooth on that scale. Using larger occulters at larger distances will make it possible to obtain edges more and more sharp.

One last point favors circular occulter for solar studies. The central part of the shadow corresponding to the Arago spot is fairly described by the only non-zero term of the Lommel series, the simple Bessel function $J_0(\pi r D/\lambda z)$. The diameter of the Arago spot can then be approximately written as $1.53\lambda/\alpha$, where $\alpha = D/z$ is the apparent diameter of the occulter as seen from the telescope's aperture. As a consequence, the Arago spot extends over a width 10 000 times smaller in the solar case than in the case of a exoplanet observation.

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References

Arenberg, J.W., Lo, A.S., Glassman, T.M. & Cash,W., 2007, C.R. Physique, 8, 438

Born, M., & Wolf, E. 2006, Principles of Optics, 7th Ed. (Cambridge University Press, 2006), 484

Evans, J.W. 1948, J.Opt.Soc.Am., 88,1083

Fort, B., Morel, C. & Spaak, G. 1978, A & A, 63, 243

Koutchmy, S. 1988, Space Sci. Rev., 47, 95

Lamy, P. , Damé, L. , Vivès, S. & Zhukov, A. , 2010, SPIE, 7731, 18

Lyot, B. 1939, Mon. Not. R. Astron. Soc., 99, 580

Purcell, J.D. & Koomen, M.J.. 1962, J.Opt.Soc.Am., 52, 596

Spitzer, L. 1962, American Sci., 50, 473