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TESTING THE RAY-TRACING CODE GYOTO

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Abstract. In the next few years, the near-infrared interferometer GRAVITY will observe the Galactic Center. Astrometric data will be obtained with an expected accuracy of 10 μas. In order to analyze those future data, we have developed a code named GYOTO to compute orbits and ray-trace images. We want to assess the validity and accuracy of GYOTO in a variety of contexts, in particular for stellar astrometry in the Galactic Center.

Keywords: Galactic Center, Black hole physics, Gravitational lensing

1 Introduction

GYOTO* (General relativitY OrbiT of Observatoire de Paris) is a ray-tracing code developed by Vincent et al. (2011). It integrates null and time-like geodesics in any analytical metrics and numerical metrics. This last property makes GYOTO a unique ray-tracing code. GYOTO can compute images and spectra for a variety of astrophysical objects, such as moving stars or accretion disks, around a Kerr black hole. Thanks to its particularity, it can also compute images or trajectories of stars orbiting exotic objects such as a boson star (Grandclément et al. 2014).

The main motivation for the development of GYOTO was to interpret the data to be obtained with the second generation VLTI instrument GRAVITY (Eisenhauer et al. 2011). This instrument will observe stars and flares orbiting Sgr A*. It will probe space-time near the central object with an expected astrometric accuracy of 10 μas. The stellar orbits measured by GRAVITY will be affected by several effects such as periastron shift and Lense-Thirring effects (Will 2008, Merritt et al. 2010). In additions, the individual astrometric measurements will be affected by relativistic effects: time delay and lensing (Bozza & Mancini 2012). All these effects need to be considered in apparent orbit model that will be fitted to the GRAVITY data, allowing to constrain the nature of Sgr A* using the GRAVITY data. Since the goal of GRAVITY is to deliver astrometry at an accuracy of 10 μas, models need to be more accurate than this value, in order to not limit the accuracy of final results. We therefore aim for a model with an astrometric accuracy of 1 μas. In this paper, we study the accuracy of GYOTO in order to determine whether this tool can be used as a foundation for a future apparent orbit model to fit the GRAVITY data. Using the star images computed by GYOTO it will be possible to get the apparent position of the star. However, the accuracy of this position will depend on the precision of the photon trajectories. Null geodesics need to be properly computed by the integrator implemented in GYOTO in order to take into account the correct bending effect. Beside, because of the 2" of field-of-view of GRAVITY, a wide range of distances between stars and Sgr A* will be possible. GYOTO has never been used in such a configuration, we need to ensure that geodesics are well computed.

We first focus on the Einstein ring radius. The aims are both to compare our numerical results with analytical study on the Einstein ring radius performed by Sereno and De Luca in 2008 (Sereno & de Luca 2008), and to check if the numerical error is sufficiently low, which means inferior or equal to 1 μas. The comparison between GYOTO and the approximation is a validation of GYOTO in the weak deflection limit (WDL), however we also have to check if this ray-tracing code is valid in the strong deflection limit (SDL). To do so, we choose to compare null geodesics computed in GYOTO and with another code named Geokerr† (Dexter & Agol 2009).

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*freely available at the URL `http://GYOTO.obspm.fr`
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2 The Einstein ring

To understand how the Einstein ring is formed, we remind basics of gravitational lensing using a Schwarzschild lens. Then, we focus on Einstein ring obtained with a Kerr black hole. In both cases we consider a static observer. The spin axis coincides with the z-axis. Spherical coordinates of the observer and the source, relative to the lens L, are noted $(r_0, \vartheta_0, \phi_0)$ and $(r_s, \vartheta_s, \phi_s)$, respectively. Without loss of generality we choose to work in the equatorial plane: $\vartheta_0 = \frac{\pi}{2}$, $\phi_0 = 0$ and $\vartheta_s = \frac{\pi}{2}$. This yields $(x_0, 0, 0)$ for the observer and $(x_s, y_s, 0)$ for the source. We note M the lens mass and a the spin of the black hole ranging from 0 (Schwarzschild black hole) to 1 (extremal Kerr black hole). In this paper, we use two different units for the distance: parsecs and geometrical units. This last unit is equal to $\frac{GM}{c^2}$, with G the Newton's constant and c the speed of light, but we will consider $G = c = 1$ and note it M.

2.1 *Schwarzschild lens*

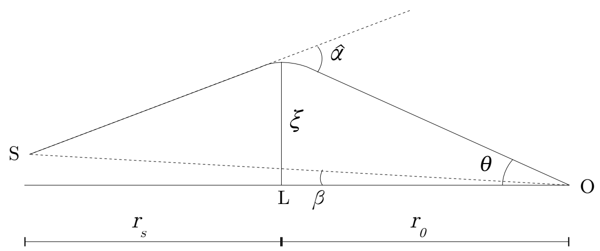

Fig. 1. Spatial projection of a Schwarzschild lensing situation: S corresponds to the source, L to the lens and O to the observer.

Using the notation of Fig.1, we can write the useful lens equation (Schneider et al. 1992):

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{r_s}{(r_s + r_0)} \hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{2.1}$$

with $\boldsymbol{\beta}$ the unlensed angular position of the source, $\boldsymbol{\theta}$ the lensed angular position of the source equal to $\boldsymbol{\xi}/r_0$ and $\hat{\boldsymbol{\alpha}}$ the deflection angle depending on the impact parameter $\boldsymbol{\xi}$. The latter angle is also called the Einstein angle whose expression is equal to $\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{2R_S}{\boldsymbol{\xi}}$ with $R_S = \frac{2GM}{c^2}$ the Schwarzschild radius. We can rewrite the lens equation as:

$$\boldsymbol{\theta}^2 - \boldsymbol{\beta}\boldsymbol{\theta} - \alpha_0^2 = 0, \tag{2.2}$$

with $\alpha_0 = \sqrt{2R_S \frac{r_s}{r_0(r_s+r_0)}}$. The magnification of the source in the lens plane is function of the lensed and unlensed angles as:

$$A = J^{-1} = \left| det \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}} \right|^{-1}. \tag{2.3}$$

A is infinite when $J = 0$. In the source plane, these positions are called caustics points (or primary caustic). For a Schwarzschild lens, the caustic is a line behind the lens starting from it and extending toward infinity (Rauch & Blandford 1994). If the source lies on the caustic line then $\beta = 0$. Thus, the solution of the lens equation is $\theta = \alpha_0$. A circle called critical curve is formed in the lens plane with a radius of α_0. Considering the source as a star the observer sees the well-known Einstein ring. The radius of the ring corresponds to the critical curve radius so we get $\alpha_0 = \theta_E$ with θ_E the Einstein ring radius. If the star does not lie on the caustic, the observer will see two images named primary and secondary images. These images are formed by lensing effects. Light rays are deviated because of the curvature of space-time by the black hole. At the caustic points, the lensed images merge into the Einstein ring.

2.2 *Kerr black hole lens*

In the Kerr black hole case, there is also a primary caustic but it is not a line anymore (Rauch & Blandford 1994, Bozza 2008). Rauch & Blandford (1994) were the first to discover that the primary caustic is a tube with an astroid (four-cusped) cross-section. At large distances the cross-section is symmetric but becomes distorted near the horizon. Besides, the closer the source to the black hole the larger the tube shifts with respect to the Schwarzschild's case. Very far from the black hole the shift is still significant but the size of the caustic (distance between the right and the left cusp of the astroid cross-section) decreases and tends to zero. To form the critical curve with this caustic, the source must cover all of the astroid cross-section.

An analytical study of the Einstein ring radius was made by Sereno and De Luca in 2006 and 2008 (Sereno & de Luca 2006, Sereno & de Luca 2008). Their analytical approximation is obtained in the WDL. In this regime, photons do not wind around the black hole which means $r_s >> R_S$ and the minimum distance between the photon and the lens r_{min_γ} must satisfy: $R_S << r_{min_\gamma}$. In this regime, the primary caustic is only shifted and keeps a symmetric shape. Because of the shift of the caustic, the critical curve is not centered on the black hole. In Sereno & de Luca 2008, the Einstein ring radius (or critical curve radius) equation is developed through a Taylor expansion of the light-like geodesics in $\varepsilon = \frac{\theta_E}{4D}$ where $D = \frac{r_s}{r_s + r_0}$ and θ_E is the Einstein ring radius. The equation presents in this paper is expressed in the equatorial plane. The radius of the Einstein ring is given by:

$$\Theta_E \simeq \theta_E \left\{ 1 + \frac{15\pi}{32}\varepsilon + \left[4\left(1 + D^2\right) - \frac{675\pi^2}{2048} \right] \varepsilon^2 + \frac{15\pi}{8}\varepsilon^3 \left[D + 4D^2 - \frac{9(272 - 25\pi^2)}{1024} - \frac{a^2}{8} \right] \right\}. \tag{2.4}$$

The left and the right radius of the critical curve are equal and depend on the spin in the third-order term in ε.

To validate GYOTO in the WDL, we estimate the Einstein ring radius with GYOTO and make a comparison with the formula (2.4). To reproduce the observational conditions of GRAVITY, we consider an observer at $r_0 = 8$ kpc from a black hole of mass M equal to $4.31 \times 10^6 M_\odot$. We also consider a source far enough from the black hole to be compliant with the domain of validity of this approximation. For each distance of the source, we estimate the error made on the Einstein ring radius. Since the goal of this paper is to determine if the accuracy of GYOTO is better than 1 μas, we only consider the maximum error of this parameter.

3 Results

3.1 Weak deflexion limit

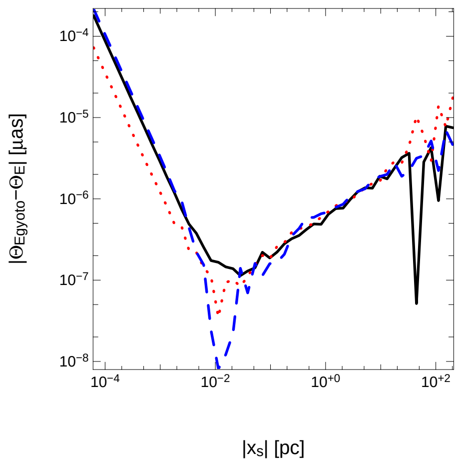

Fig. 2. Absolute difference between the analytical approximation Θ_E and Einstein ring radius measured with GYOTO. The types of line denote different values of the spin: 0.2 in solid, 0.5 in dotted and 0.9 in dashed.

On Fig.2 we present the absolute difference between the analytical approximation (2.4) and Einstein ring radius measured with GYOTO, obtained for three different spins. For all the range of the parameter, the differences are always extremely small ($\sim 10^{-2}\mu$m). On the plot two different regimes can be observed. For small x_s, the curve is marked by a smooth, power-law decrease: GYOTO and the numerical approximation agree better and better for larger and larger values of x_s. After reaching a minimum, the curve raises again, which a much more noisy appearance. This is due to the fact that, for small x_s, GYOTO is better than the analytical approximation. The difference between the two traces the order of the approximation. On the other hand, for large values of x_s, the approximation wins over GYOTO and the difference is dominated by the numerical error of GYOTO. The maximal errors of the parameter evaluated with GYOTO, and for each spin, are all smaller than $10^{-3}\mu$as. The ray-tracing code is very accurate, even for sources far behind the black hole (e.g. $\delta_{\Theta_{Egyoto}} = 1.6^{-4}\mu$as at 200 parsecs).

The requirement on accuracy ($\leq 1\mu$as) is largely met in the weak field regime. However, an equivalent test is necessary in the SDL regime.

3.2 Strong deflexion limit

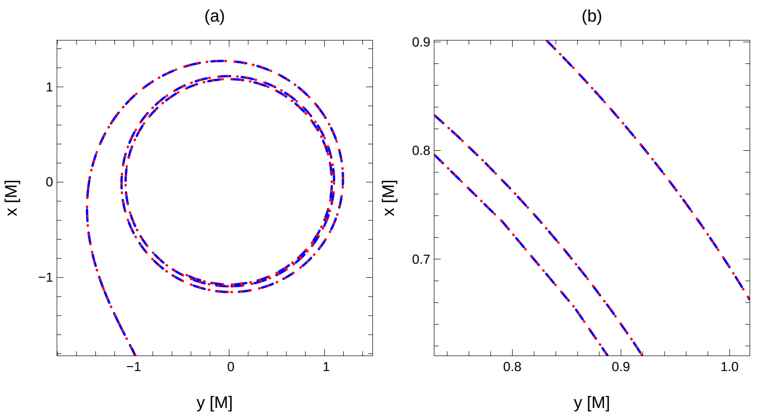

Fig. 3. Null geodesics computed with GYOTO (dotted) and Geokerr (dashed). We consider a spin of 0.998 and a photon launched from the center of the screen. (b) is a zoom of geodesics plotted on (a): δ_x and δ_y on (b) is about $4 \times 10^{-4} M$.

The aim of this subsection is to check if null geodesics computed with GYOTO in the SDL are accurate enough. To do so, we decided to compare photon trajectories computed with GYOTO with those computed with the ray-tracing code Geokerr. Contrary to GYOTO, Geokerr computes photon coordinates semi-analytically reducing the equations of motion expressed in the Hamiltonian formulation to Carlson elliptic integrals.

The comparison is made using the same observer coordinates and black hole parameters as before. We evaluate null geodesics for three different values of the spin (0, 0.5 and 0.998) and we consider photons launched from the center of the observer screen ($\alpha = \delta = 0$). We first compute the geodesics with Geokerr and get the dates of each point of the photon trajectory. Then, to get the null geodesics with GYOTO, we interpolate the positions of photons with our ray-tracing code considering these dates.

The difference between positions evaluated with GYOTO and those evaluated with Geokerr are smaller than $10^{-3}\mu$as for a spin equal to 0.2, 0.5 or 0.998. This shows a very good consistency between the two ray-tracing codes. Even for a photon which is not launched from the center of the screen (α =1.2 μpc), with $a = 0.998$, we find a very small error ($\sim 6 \times 10^{-4} M$). An example of computed null geodesics with both codes are shown on Fig.3.

4 Conclusions

The Galactic Center is a unique laboratory to observe stars close enough to a compact object to test General Relativity. Thanks to GRAVITY it will be possible to measure astrometric positions of stars orbiting Sgr A* with an expected astrometric precision of 10 μas. We have shown that GYOTO is extremely accurate even in complex configurations. For the purpose of the interpretation of the future astrometric positions observed by GRAVITY, GYOTO is accurate enough to model star trajectories and fit the GRAVITY data.

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