SF2A 2016
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WHY CURVATURE RADIATION IN NEUTRON-STAR MAGNETOSPHERES SHOULD BE TREATED IN THE FRAMEWORK OF QUANTUM ELECTRODYNAMICS

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Abstract. Curvature radiation is a key phenomenon in pulsar and magnetar magnetospheres. It is classically conceptually very close to synchrotron radiation, however we will show that in ultra-relativistic very-high-magnetic-field environments, the same approximations that lead to its use are also leading quickly to a potential quantized regime where the classical theory may fail. We explain in some details these caveats and give an outline of a quantum-electrodynamics treatment. We show that the internal consistency of the theory of curvature radiation is improved, and some interesting effects due to spin-flip transitions may occur.

Keywords: Neutron stars, pulsars, magnetars, synchrotron, curvature radiation, quantum electrodynamics

1 Introduction

Electron and positron states with very low momentum perpendicular to the magnetic field have been of interest in the field of rotating neutron-star magnetospheres almost since their discovery in 1968 Hewish et al. (1968). Indeed, the community soon realized that the extremely intense rotating magnetic fields of those magnetospheres, ranging from $\sim 10^4$ Teslas at the surface of old millisecond pulsars to $\sim 10^{11}$ Teslas at the surface of some magnetars with a typical $\sim 10^8$ Teslas Viganò et al. (2015), could generate extremely large electric-potential gaps along the open magnetic-field lines (see e.g. Arons (2009) for a review) which in turn accelerate charged particles to energies only limited by radiation reaction. It is believed that these magnetospheres are mostly filled with electrons and positrons resulting from a cascade of pair creations : pairs are created by quantum-electrodynamics processes involving gamma rays, and in turn radiate their kinetic energy in gamma-rays that make other pairs. The process of radiation is that of an accelerated charge that inspirals around a curved magnetic field. Because the magnetic field $\vec{B}$ is so intense, radiation reaction quickly forces the particle to follow very closely the field line. It follows that electrons and positrons are believed to radiate mostly because of their motion along the curved field line rather than perpendicular to it. Such motion and radiation are described either by the synchro-curvature regime or the curvature regime (Ruderman & Sutherland 1975), depending on whether the residual perpendicular motion is taken into account or neglected.

We show in section 2 the treatment of curvature radiation within the framework of classical electrodynamics and we demonstrate in section 3 that in neutron-star magnetospheres it quickly leads to apply the classical theory when momentum is already significantly quantized. In section 4 we give an outline of the theory of quantum curvature radiation.

2 Consistency of the theory of curvature radiation

In the ultra-relativistic regime, the classical treatment of curvature radiation is fundamentally the same as that of the synchrotron radiation (Jackson 1998). In the extreme environments surrounding pulsars and magnetars we are interested in, the ultra-relativistic approximation is always appropriate. There are mostly two reasons to this similarity. First in the ultra-relativistic regime the beam of emitted light is very collimated with a typical angle $\sim 1/\gamma$ where $\gamma \gg 1$ is the relativistic Lorentz factor. It follows that the light finally catched by an observer

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was necessarily emitted on a very small portion of the trajectory, which in turn needs only be locally circular. The second reason is the neglecting of radiation back-reaction on the motion of the particle, that allows to treat motion and radiation in a completely separated way. Therefore, the fact that the path be curved because of a magnetic field or any other source does not matter. Finally, classical synchrotron radiation appears as a particular case of curvature radiation.

In the context of pulsar magnetospheres, the path followed by electrons and positrons is assumed to be a magnetic-field line, to which one can add the $\vec{E}\times\vec{B}/B^2$ drift, where $\vec{E}$ is the electric field, $\vec{B}$ the magnetic field. This motion is not physical, since it does not follow the usual helicoidal solution of the motion of a charged particle in a magnetic field. In the case of a particle following a magnetic-field line, the Lorentz force $\vec{v}\times\vec{B}$ is the only force acting on the particle and is exactly zero. Therefore, a charged particle cannot follow a magnetic-field line without turning, even slightly, around the field.

However, one assumes such a path as a result of the extreme radiation reaction undergone by a charged particle. Let's take a few numbers that we will consider typical of a pulsar polar cap gap. We consider an accelerating electric field of intensity E, assumed collinear to the magnetic field of intensity B. Close to the neutron star surface, a dipolar magnetic field locally has a radius of curvature ρ of the order of magnitude of the neutron star radius. Assuming the electric field is given by a force-free condition around a star rotating at Ω_* (see e.g. (Arons 2009)) one has

$$\rho \sim 10^4\mathrm{m}, \Omega_* \sim 1\mathrm{s}, B \sim 10^8\mathrm{Teslas}, E = \Omega_* R_* B \sim 10^{12}\mathrm{V/m}. \tag{2.1}$$

In these conditions, an electron or a positron accelerates almost instantaneously, that is on a length scale much shorter than the size of the gap, until radiation reaction balances the electric field. If one assumes that losses are only due to curvature radiation then the radiated power is $\propto \Omega_c^2\gamma^4$, with $\Omega_c = c/\rho$ the pulsation of an imaginary circular trajectory of radius ρ traveled at the speed of light c, the equilibrium Lorentz factor is (Viganò et al. 2014)

$$\gamma_{\max} = \left(\frac{3}{2}\frac{4\pi\epsilon_0 E\rho}{e}\right) \sim 2\cdot 10^7 E_{12}^{1/4}\rho_4^{1/4}, \tag{2.2}$$

with $-e$ the charge of the electron and ϵ_0 the vacuum electric permittivity (in international system units), and we use the notation $X_n = 10^{-n}X$. If the particle bears an initial momentum perpendicular to the magnetic field, it can only be small compared to the longitudinal momentum, because in the opposite case the dominant losses are from synchrotron which follows the same scaling law but with a pulsation Ω_s much larger than the curvature pulsation Ω_c

$$\Omega_s = \frac{eB}{\gamma m} \sim 10^{12} B_8\gamma_7^{-1} \gg \Omega_c \sim 10^4\rho_4^{-1}, \tag{2.3}$$

resulting in a dissipation $10^{16}B_8^2\gamma_7^{-2}\rho_4^{-2}$ times more efficient for the same Lorentz factor. That is why an electron or positron cannot have a perpendicular momentum even comparable to its longitudinal momentum. This is the justification of curvature radiation, that assumes that all perpendicular momentum is dissipated.

However, a small perpendicular component must remain. It suggests to compute the radiation of a particle following an helix in the approximation of a small pitch-angle α, approximation usually called synchro-curvature radiation (see e.g. Cheng & Zhang (1996), Harko & Cheng (2002), Viganò et al. (2014) or Kelner et al. (2015)). One understands that curvature radiation, however based on an unphysical path, is the natural mathematical limit when $\alpha \to 0$ of synchro-curvature radiation. If one assumes that thanks to relativistic beaming the radiation-reaction force is directed in the exact opposite direction to the velocity of the particle and that radiation-reaction balances the electric field, one can quickly obtain the evolution of the pitch-angle of the particle (See e.g. Viganò et al. (2014))

$$\sin\alpha = \sin\alpha_0 \exp\left(-\frac{t}{\tau_\alpha}\right) \tag{2.4}$$

where α_0 is the initial pitch angle and

$$\tau_\alpha = \frac{\gamma_{\max} mc}{eE} \sim 2\cdot 10^{-8}\gamma_{\max 7}E_{12}^{-1}\mathrm{s} \tag{2.5}$$

is the characteristic decay time.

As a consequence, the classical theory predicts an arbitrary decay of the pitch angle on very short distances.

3 Limit of the classical theory

The quantum theory of a relativistic electron in an uniform magnetic field was derived by several authors (Huff (1931), Johnson & Lippmann (1949), Melrose & Parle (1983), Sokolov & Ternov (1968)) . It is applicable to all spin 1/2 particles. However we consider an electron to simplify the presentation. The electron is characterized by the quantified angular momentum around the magnetic field, quantified momentum parallel to the field and two possible spin orientations. The energy of the particle is the sum of the squared perpendicular and longitudinal momenta plus the squared rest mass energy

$$E = \sqrt{m^2c^4 + \underbrace{\hbar\omega_c mc^2 n}_{\text{Perpendicular momentum}} + \underbrace{(cp_{\parallel})^2}_{\text{Parallel momentum}}} \tag{3.1}$$

where n is an integer quantifying the perpendicular momentum, $\omega_c = eB/m$ is the cyclotron pulsation and $p_{\parallel}$ the parallel momentum. The two spin orientations are degenerate with respect to the energy. In quantum theory, the levels are quantified by the number n. They are sometimes referred to as Landau levels; we call them perpendicular levels. Transitions between perpendicular levels are at the origin of synchrotron radiation.

The classical limit of a quantum theory means, in particular, that the quantized step of a given quantity is negligible compared the value of this quantity. In the case of perpendicular momentum it means that $1/\sqrt{n} \ll 1$. The step $\sqrt{\hbar\omega_c mc}$ increases with the magnetic-field intensity. Transitions between perpendicular levels then become quasi-continuous, and one finds the classical limit of synchrotron radiation.

The decay of pitch-angle calculated in the previous section corresponds in the quantum theory to the decay of n. If one extrapolates a little the theory in a uniform field to a curved magnetic field, one understands that the limit of curvature radiation then corresponds to $n = 0$. This means a regime in which perpendicular momentum cannot be treated classically. But is this regime ever reached ? For an ultra-relativistic particle most of the energy is in the longitudinal term $cp_{\parallel} \sim \gamma mc^2$, and one can estimate the pitch angle of the first perpendicular state as

$$\alpha_1 \simeq \frac{1}{\gamma}\sqrt{\frac{\hbar\omega_c}{mc^2}} \sim 10^{-8} B_8^{1/2} \gamma_7^{-1}. \tag{3.2}$$

Equation 2.5 implies that this pitch angle would be reached in barely 10 meters if the classical theory is correct. However, as we have seen, we are out of the domain of validity of the classical theory.

4 Sketch of a quantum theory

Here we intend to outline the main physical ideas behind a quantum theory of curvature radiation. Detailed calculations and results, including spectra, will be detailed in two upcoming papers (in preparation) and to some extent in Voisin et al. (2016, in press).

The first interesting point, from a qualitative point of view, is that it is possible to make synchrotron-like radiation within the quantum formalism in a way that is relatively independent of the physical ingredients (or classically "forces") that cause the motion, or here the wave function. An - even locally - circular path translates in the vocabulary of quantum mechanics in rotation invariance around the axis of the circle. Therefore, our wave-function must be a proper state of the angular momentum operator around this axis, as it is also the generator of rotations (see e.g. Le Bellac (2003)). Of course, if this is our only requirement. We impose the corresponding arbitrary Hamiltonian $\hat{H}$ which in this case is merely proportional to the angular momentum operator $\hat{J}$,

$$\hat{H} = \Omega \hat{J} \tag{4.1}$$

, where for curvature radiation in the utlra-relativistic limit, $\Omega = \Omega_c = c/\rho$. Proper values are then given by $E = \hbar\Omega l$ where l is a half-integer. This kind of Hamiltonian is found as a limit when the perpendicular term dominates in equation 3.1, for example. One last thing is that our theory can only be one-dimensional since our wave-function is cylindrically symmetric but extends to infinity in space. The one dimension can be parametrized by the angle around the axis of the cylinder. Applying a first-order perturbation theory, sometimes called Fermi golden rule, to our Hamiltonian with the usual interaction Hamiltonian of quantum electrodynamics one finds transitions between proper wave-functions that are identical to the classical curvature radiation of pulsation Ω, up to a prefactor due to the lack of relevant 3-dimensional confinement.

Our theory therefore needs to implement two rotation invariances, one around the axis of the main circular trajectory, and locally around the tangent to this trajectory. Although not intuitive, the two corresponding rotation operators commute with each other allowing an independent treatment. We need a magnetic field to confine the particle on such a trajectory, a possible choice is a field with concentric field lines around the main axis. As known from the uniform-field theory the characteristic confinement length around the magnetic field is given by $\lambda = \left(\frac{2\hbar}{eB}\right)^{1/2}$. In the case of pulsar fields, this length is very small compared to the radius of curvature allowing to expand equations in a small parameter

$$\epsilon = \frac{\lambda}{\rho} \sim 10^{-16} B_8^{-1/2} \rho_4^{-1} \tag{4.2}$$

. At zeroth order, the two previous rotation operators commute with the Hamiltonian and are therefore observables. We need no more. Thus we can separate the two rotations in our physical model. The proper energies are

$$E = \sqrt{m^2c^4 + \underbrace{\hbar\omega_c mc^2 n}_{\text{Perpendicular momentum}} + \underbrace{(\hbar\Omega l)^2}_{\text{Parallel momentum}}}, \tag{4.3}$$

where the only difference with 3.1, quite satisfactorily, is that the parallel momentum is now replaced by the angular momentum around the main axis. Ω and l are defined as in the toy model of the previous paragraph. One remarks that when motion parallel to the field dominates, this term dominates the energy and we find the same expression as in our toy model at main order.

Classical curvature radiation is recovered in this model by computing transitions between states with $n = 0$ in the ultra-relativistic limit $l \gg 1, \hbar\Omega l \gg mc^2$. The photon energy is taken from a variation of l. In this class of states the particle as close as possible to the field line, with a spreading of only $\sim \lambda$. Additionally one shows that these states correspond to a null orbital angular momentum. What was unphysical in the classical theory becomes possible in quantum mechanics. However, there is a trick : the spin of this perpendical "fundamental" state is oriented backward the magnetic field giving a negative total angular momentum of $-1/2\hbar$. Thus, the particle holds on the magnetic field due to the interaction between its spin and the field.

It is interesting to generalize a little the definition of curvature radiation to incorporate different spin orientations. As a result of spin-orbit coupling, this can only be done by taking into account transitions to the first excited perpendicular states. This can be considered a first step towards quantum synchro-curvature radiation or spin-flip curvature radiation. It proves to be not negligible in typical pulsar, young pulsar and potentially magnetar environments.

5 Conclusions

An electron in a typical pulsar or magnetar magnetosphere falls very quickly in the synchro-curvature low-pitch-angle regime due to radiation reaction on the particle. Curvature radiation is the natural mathematical and physical limit of synchro-curvature radiation within classical electrodynamics. However, in these very high magnetic fields the momentum perpendicular to the magnetic field very quickly becomes comparable to its quantization step. A quantum theory of the motion of an electron in a curved magnetic field can be devised as a generalization of the existing theory in a uniform magnetic field. In this case, classical curvature radiation survives in a more consistent way, since states with null orbital momentum become physical, thanks to spin-magnetic-field interaction. If one allows spin-flip to happen then new transitions appear that can be not negligible in neutron-star environments.

References

Arons, J. 2009, in Neutron Stars and Pulsars, ed. W. Becker, Astrophysics and Space Science Library No. 357 (Springer Berlin Heidelberg), 373–420

Cheng, K. S. & Zhang, J. L. 1996, The Astrophysical Journal, 463, 271

Harko, T. & Cheng, K. S. 2002, Monthly Notices of the Royal Astronomical Society, 335, 99

Hewish, A., Bell, S. J., Pilkington, J. D. H., Scott, P. F., & Collins, R. A. 1968, Nature, 217, 709

Huff, L. D. 1931, Physical Review, 38, 501

Jackson, J. D. 1998, Classical Electrodynamics, 3rd Edition

Johnson, M. H. & Lippmann, B. A. 1949, Physical Review, 76, 828

Kelner, S. R., Prosekin, A. Y., & Aharonian, F. A. 2015, The Astronomical Journal, 149, 33, arXiv: 1501.04994

Le Bellac, M. 2003, Physique quantique (Les Ulis; Paris: EDP Sciences)

Melrose, D. B. & Parle, A. J. 1983, Australian Journal of Physics, 36, 755

Ruderman, M. A. & Sutherland, P. G. 1975, The Astrophysical Journal, 196, 51

Sokolov, A. A. & Ternov, I. M. 1968, Synchrotron Radiation (Akademie-Verlag)

Viganò, D., Pons, J. A., Miralles, J. A., & Rea, N. 2015, eprint: arXiv:1501.06735, 90–101

Viganò, D., Torres, D. F., Hirotani, K., & Pessah, M. E. 2014, Monthly Notices of the Royal Astronomical Society, 447, 1164

Voisin, G., Bonazzola, S., & Mottez, F. 2016, in press, in AIP Conference Proceedings (AIP Publishing)