SF2A 2024
M. Béthermin, K. Baillié, N. Lagarde, J. Malzac, R.-M. Ouazzani, J. Richard, O. Venot, A. Siebert (eds)

MEASURING LORENTZ INVARIANCE VIOLATIONS WITH GRAVITATIONAL WAVES AND THE SME FORMALISM

S. Aoulad Lafkih[1], M.-C. Angonin[1], C. Le Poncin-Lafitte[1] and N. A. Nilsson[1, 2]

Abstract. In this presentation, we study Lorentz violation through the generation of gravitational waves. These results coupled with the new gravitational waves surveys like the one conducted by LISA, will allow us to put state-of-the-art constraints on these symmetries violations. In order to do so, we select a Lorentz-violating extension of General Relativity in the pure gravity sector, directly taken from the Standard Model Extension (SME) formalism. We derive a wave equation for the metric perturbation from the modified Einstein equation, and attempt to solve it with a Post-Minkowskian method. In this article we outline the benefits and difficulty of this method and give some preliminary results for the waveforms.

Keywords: General Relativity, Lorentz invariance, Post-Minkowski, Standard Model Extension

1 Introduction

Many theories aiming to bridge the gap between General Relativity (GR) and Quantum Field Theory (QFT) invoke the presence of fields breaking the Lorentz invariance, which is a fundamental principle of Einstein's theory of Relativity (Einstein (1916)). But GR is extremely efficient on most scales, so any deviations from it must be very small. In order to more effectively quantify these weak departure from the classical regime of General Relativity, the Standard Model Extension (SME) was created in the early 2000's (Kostelecký (2004)). Its scope encompasses a large range of symmetries violations in General Relativity, as well as QFT. In the pure gravity sector, it encodes all possible violations of the Lorentz violations from first principles, directly in the action. Many terms have been considered in the literature, and they can be arranged by growing mass order. Here we consider a simple model with the lowest mass order of 4, originally proposed in Bailey & Kostelecký (2006).

An interesting range of observations and experiments can be used to constrain the SME coefficients (Kostelecký & Russell (2011)), in the gravity sector, but also in the particle or electromagnetic sectors. One such observation is gravitational waves, its peculiarity residing in the fact it probes a regime of strong gravity where we might expect stronger deviations from the standard predictions of GR. This new astronomical messenger can also be studied along light observations for a more thorough information extraction. There has already been some constraints on the effect of Lorentz violation on the propagation and birefregence of gravitational wave in the vacuum of space (Haegel et al. (2023)), and especially through one event : GW 170817. This event was of particular significance as it was detected alongside its electromagnetic counterpart. Despite its briefness, the event allowed astrophysicist to perform a lot of tests on the propagation of electromagnetic and gravitational waves with respect to one another (Monitor et al. (2017)).

The LISA mission should also be a treasure trove of new astrophysical observations in this regard (Barausse et al. (2020)). The launch is planned for 2037 and should observe gravitational waves from space for a duration of at least 4 years. Because its frequency range is much lower than LIGO and VIRGO, from 0.1 mHz to 0.1 Hz,

[1] SYRTE, Observatoire de Paris, PSL Research University, CNRS, Sorbonne Universités, UPMC Univ. Paris 06, LNE, 61 avenue de l'Observatoire de Paris, 75014 Paris, France
[2] Cosmology, Gravity and Astroparticle Physics Group, Center for Theoretical Physics of the Universe, Institute for Basic Science, Daejeon 34126, Korea

it should be able to observe many galactic binary systems that are still far from coalescence. So in principle we should be able to access a lot of different quasi-monochromatic signals for longer period of times. This work is a preparatory step in order to apply constraints on the SME coefficients through the data of the future LISA mission.

In section 2 we show the wave equation linked to our model and comment on the source terms. In section 3, we discuss the inverse d'Alembertian used to solve this differential wave equation. In section 4 we show some particular solutions of the wave equations and discuss their structure. And finally, in section 5 we discuss some ways of improving and continuing this project.

2 Obtaining a wave equation

2.1 SME formalism

The SME formalism used for this project is from Bailey & Kosteleckỳ (2006). It shows terms of mass order 4 directly parametrised in the action S as such :

$$S = \int \sqrt{-g}\left(R\left(1-u\right) + s^{\mu\nu}R_{\mu\nu} + t^{\lambda\kappa\mu\nu}C_{\lambda\kappa\mu\nu}\right)\mathrm{d}^4x. \tag{2.1}$$

In GR the sole term is the Einstein-Hilbert Lagrangian : $L_{EH} = \sqrt{-g}R$, with g being the determinant of the metric, and R the Ricci scalar. To this term, Bailey and Kostelecký adjoined another term with objects breaking Lorentz invariance : u, $s^{\mu\nu}$ and $t^{\lambda\kappa\mu\nu}$. These tensors are contracted with tensors of General Relativity : the Ricci tensor $R_{\mu\nu}$ and the Weyl tensor $C_{\lambda\kappa\mu\nu}$. The SME tensors are traceless and possess some indices symmetry in correlation to the metric object they are contracted to : $s^{\mu\nu}$ has the indices symmetries of the Ricci tensor and $t^{\lambda\kappa\mu\nu}$ of the Weyl tensor.
After a perturbative treatment, Bailey and Kostelecký find the following linearised Einstein equation :

$$\tilde{G}_{\mu\nu} = \bar{u}G_{\mu\nu} + \eta_{\mu\nu}\bar{s}^{\alpha\beta}R_{\alpha\beta} - 2\bar{s}^{\alpha}{}_{(\mu}R_{\nu)\alpha} + \frac{1}{2}\bar{s}_{\mu\nu}R + \bar{s}^{\alpha\beta}R_{\alpha\mu\nu\beta}. \tag{2.2}$$

$\tilde{G}_{\mu\nu}$ is the linearised Einstein tensor composed with the metric perturbation $h^{(1)}_{\mu\nu}$ generated by the Lorentz violating terms. Whereas the other metric object without "~" are the linearised (respectively) Einstein tensor, Ricci tensor, Ricci scalar and Riemann tensor connected to the metric perturbation of General Relativity $h^{(0)}_{\mu\nu}$.

2.2 The wave equation

Once we impose a trace-reversal on the metric perturbations ($\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$) in (2.2), and a harmonic gauge ($\partial^{\mu}\bar{h}_{\mu\nu} = 0$), we recover the following wave equation :

$$\begin{aligned}\Box\bar{h}^{(1)}_{\mu\nu} = \Box&\left[\bar{u}\bar{h}^{(0)}_{\mu\nu} + \eta_{\mu\nu}\bar{s}^{\alpha\beta}\bar{h}^{(0)}_{\alpha\beta} - 2\bar{s}^{\alpha}{}_{(\mu}\bar{h}^{(0)}_{\nu)\alpha} + \frac{1}{2}\bar{s}_{\mu\nu}\bar{h}^{(0)}\right]\\ &- 2\bar{s}^{\alpha\beta}\left(\partial_{\mu}\partial_{[\nu}\bar{h}^{(0)}_{\beta]\alpha} + \partial_{\alpha}\partial_{[\beta}\bar{h}^{(0)}_{\nu]\mu}\right) - \frac{1}{2}\left(\bar{s}_{\nu}{}^{\beta}\partial_{\mu\beta}\bar{h}^{(0)} - \bar{s}^{\alpha\beta}\eta_{\mu\nu}\partial_{\alpha\beta}\bar{h}^{(0)} + \bar{s}^{\alpha}{}_{\mu}\partial_{\alpha\nu}\bar{h}^{(0)}\right).\end{aligned} \tag{2.3}$$

Now that we have this form for the source terms, we introduce a modelisation of the linearised gravitational wave from General Relativity. We only consider the first and stronger term (in a PN expansion), the quadrupole formula (Hobson et al. (2006)) (throughout this paper we take the following units $G = c = 1$) :

$$
\begin{aligned}
\bar{h}_{00}^{(0)}\left(ct,\vec{x}\right) &= 4\frac{M}{r}, \\
\bar{h}_{i0}^{(0)}\left(ct,\vec{x}\right) &= 0, \\
\bar{h}_{ij}^{(0)}\left(ct,\vec{x}\right) &= -2\frac{\ddot{I}_{ij}\left(ct-r\right)}{r}, \\
\bar{h}^{(0)}\left(ct,\vec{x}\right) &= 2\frac{2M+\ddot{I}\left(ct-r\right)}{r}.
\end{aligned}
\tag{2.4}
$$

Where, for clarity's sake we have introduced the following notations :

- $|\vec{x}| = r$
- M is the total mass of the system, defined as $M = \int_S T_{00} d^3\vec{y}$ (S is a volume encompassing the whole system)
- I_{ij} is the mass quadrupole and $I = \delta^{ij} I_{ij}$ is its trace, $I^{ij} = \int_S T^{00} x^i x^j \mathrm{d}^3 x$. This mass multipole is a function of the retarded time $t - r$ only.
- The dots over the quadrupole denote a derivation with respect to the time coordinate t.

Introducing (2.4) in (2.3) we find for the spatial indices (ij) :

$$
\begin{aligned}
\Box\bar{h}_{ij}^{(1)} =& -12\bar{s}^{00}\hat{n}_{ij}\frac{M}{r^3} + 2\bar{s}^{ab}\left[\hat{n}_{ij}\left(3\frac{\ddot{I}_{ab}}{r^3} + 3\frac{\dddot{I}_{ab}}{r^2} + \frac{\ddddot{I}_{ab}}{r}\right) + \frac{\delta_{ij}}{3}\frac{\ddddot{I}_{ab}}{r}\right] \\
& + 2\bar{s}^{00}\frac{\ddddot{I}_{ij}}{r} - 4\bar{s}^{0a}n_a\left(\frac{\dddot{I}_{ij}}{r^2} + \frac{\ddddot{I}_{ij}}{r}\right) + 2\bar{s}^{ab}\left[\hat{n}_{ab}\left(3\frac{\ddot{I}_{ij}}{r^3} + 3\frac{\dddot{I}_{ij}}{r^2} + \frac{\ddddot{I}_{ij}}{r}\right) + \frac{\delta_{ab}}{3}\frac{\ddddot{I}_{ij}}{r}\right] \\
& + 4\bar{s}^{0a}n_{(j}\left(\frac{\dddot{I}_{i)a}}{r^2} + \frac{\ddddot{I}_{i)a}}{r}\right) - 4\bar{s}^{ab}\left[\hat{n}_{a(j}\left(3\frac{\ddot{I}_{i)b}}{r^3} + 3\frac{\dddot{I}_{i)b}}{r^2} + \frac{\ddddot{I}_{i)b}}{r}\right) + \frac{\delta_{a(j}}{3}\frac{\ddddot{I}_{i)b}}{r}\right] \\
& - 2\bar{s}_{(j}{}^{0}n_{i)}\left(\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) + 2\bar{s}_{(j}{}^{a}\left[\hat{n}_{i)a}\left(3\frac{2M+\ddot{I}}{r^3} + 3\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) + \frac{\delta_{i)a}}{3}\frac{\ddddot{I}}{r}\right] \\
& + \delta_{ij}\left[-\bar{s}^{00}\frac{\ddddot{I}}{r} + 2\bar{s}^{0a}n_a\left(\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) - \bar{s}^{ab}\left(\hat{n}_{ab}\left(3\frac{2M+\ddot{I}}{r^3} + 3\frac{\dddot{I}}{r^2} + \frac{\ddddot{I}}{r}\right) + \frac{\delta_{ab}}{3}\frac{\ddddot{I}}{r}\right)\right]
\end{aligned}
\tag{2.5}
$$

The differential equation for the temporal indices (00) and mixed $(0j)$ have the same kind structure and source terms, albeit shorter. The $\hat{n}_L$ tensors are multiplication of l unitary direction tensors $n^i = \frac{x^i}{r}$ symmetrised and "trace-freed" (STF) over all of their indices.

3 The inverse d'Alembertian

The classic inverse d'Alembertian found in electromagnetism or in Hobson et al. (2006) is especially suited when the source term is either negligible far-away from the source, or when it is a function the dirac distribution. It is not the case here, the source terms spread everywhere in space.
However, the formulas in Blanchet & Damour (1986) are especially suited to this problem as they give an inverse d'Alembertian for certain terms :

$$
\begin{aligned}
\Box_R^{-1}\left(\hat{n}_L r^{B-k} F(t-r)\right) &= \frac{1}{D(B-k)}\int_{-\infty}^{t'-r'} \mathrm{d}s F(s)\hat{\partial}'_L\left[\frac{\left(t'-r'-s\right)^{B-k+l+2} - \left(t'+r'-s\right)^{B-k+l+2}}{r'}\right], \\
D(B-k) &= 2^{B-k+3}(B-k+2)(B-k+1)...(B-k+2-l),
\end{aligned}
\tag{3.1}
$$

where $\hat{n}_L$ is a STF unitary direction tensor, k is strictly positive and F is a function of the retarded time $t-r$ only (in the formula the term on the right is evaluated at $(t'-r', \vec{x}')$). This is exactly the form of the source terms in (2.5).

4 The particular solutions

The particular solution space is investigated by using (3.1) on the source terms in (2.5). We find the following expressions the particular solution space of $\Box\bar{h}_{ij}^{(1)}$:

$$
\begin{aligned}
\Box^{-1}\left[\Box\bar{h}_{ij}^{(1)}\right] =& 2\bar{s}^{00}\hat{n}_{ij}\frac{M}{r} - \bar{s}^{ab}\left[\hat{n}_{ij}\left(\frac{\ddot{I}_{ab}}{r} + \dddot{I}_{ab}\right) + \frac{\delta_{ij}}{3}\dddot{I}_{ab}\right] \\
& - \bar{s}^{00}\dddot{I}_{ij} + 2\bar{s}^{0a}n_a\dddot{I}_{ij} - \bar{s}^{ab}\left[\hat{n}_{ab}\left(\frac{\ddot{I}_{ij}}{r} + \dddot{I}_{ij}\right) - \frac{\delta_{ab}}{3}\dddot{I}_{ij}\right] \\
& - 2\bar{s}^{0a}n_{(j}\dddot{I}_{i)a} + 2\bar{s}^{ab}\left[\hat{n}_{a(j}\left(\frac{\ddot{I}_{i)b}}{r} + \dddot{I}_{i)b}\right) + \frac{\delta_{a(j}}{3}\dddot{I}_{i)b}\right] \\
& + \bar{s}_{(j}{}^{0}n_{i)}\dddot{I} - \bar{s}_{(j}{}^{a}\left[\hat{n}_{i)a}\left(\frac{2M+\ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{i)a}}{3}\ddddot{I}\right] \\
& - \frac{\delta_{ij}}{2}\left[-\bar{s}^{00}\dddot{I} + 2\bar{s}^{0a}n_a\dddot{I} - \bar{s}^{ab}\left(\hat{n}_{ab}\left(\frac{2M+\ddot{I}}{r} + \dddot{I}\right) + \frac{\delta_{ab}}{3}\dddot{I}\right)\right].
\end{aligned}
\tag{4.1}
$$

We remark that the particular solution is not spherically symmetric, contrary to $\bar{h}_{ij}^{(0)}$, by the presence of the unitary directional multipoles $\hat{n}_L$. We also observe that the fastest decreasing terms do so at the same rate as $\bar{h}_{ij}^{(0)}$, as $1/r$, and that there are some constant terms that do not depend on the retarded time at all. The most surprising terms are those that do not decrease at all with respect to r, they seem un-physical and would constitute a strong no-go theorem against the SME coefficients they are contracted with, if they do survive the passage to observables.

5 Conclusion

This approach has given interesting results in the form of particular solutions to the wave equation in (2.3). It should be completed by imposing the harmonic gauge on our final solution through a precise choice of a homogeneous solution, and a calculation of observables. It could then further be improved by taking a more complete model for $\bar{h}_{\mu\nu}^{(0)}$, including the complete serie of mass multipoles. The number of SME coefficients could then be expanded with the Einstein-Lifshitz formulation of gravity in order to get rid of the weak-field approximation. With respect to the LISA data, the final prediction of this project should be included in the LISA pipeline so that a parameter estimation can be performed on the SME coefficients as soon as new signal is observed.

I would like to thank the organizers of the SF2A for this great event, and for agreeing to give me a talk.

References

Bailey, Q. G. & Kostelecký, V. A. 2006, Physical Review D—Particles, Fields, Gravitation, and Cosmology, 74, 045001

Barausse, E., Berti, E., Hertog, T., et al. 2020, General Relativity and Gravitation, 52, 1

Blanchet, L. & Damour, T. 1986, Philosophical transactions of the royal society of London. Series a, mathematical and physical sciences, 320, 379

Einstein, A. 1916

Haegel, L., O'Neal-Ault, K., Bailey, Q. G., et al. 2023, Physical Review D, 107, 064031

Hobson, M. P., Efstathiou, G. P., & Lasenby, A. N. 2006, General Relativity: An Introduction for Physicists (Cambridge University Press)

Kostelecký, V. A. 2004, Physical Review D, 69, 105009

Kostelecký, V. A. & Russell, N. 2011, Reviews of Modern Physics, 83, 11

Monitor, F. G.-r. B., Collaboration, L. S., Collaboration, V., et al. 2017, arXiv preprint arXiv:1710.05834