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A NEW LAGRANGIAN STOCHASTIC CODE FOR STELLAR CONVECTION

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Abstract. Convection is ubiquitous in the interior of stars, and affects stellar structure and evolution through large-scale transport of energy, angular momentum and chemical elements. Realistic inclusion of convection in stellar evolution models constitutes one of the major challenges in stellar physics, in particular where interface layers between convective and radiative regions are concerned. With the upcoming launch of the PLATO mission, overcoming the obstacle that this model defficiency poses to asteroseismology becomes a crucial and timely challenge, and one that requires our theoretical knowledge of stellar interiors to keep up with the developments of our observational capabilities.

The treatment of convection in stellar evolution models usually involves Mixing Length Theory. While it is useful in describing the bulk of convective regions, it does not allow to properly describe convective/radiative interface layers, and is also notoriously bad at describing the interplay between waves and convection that is at the heart of seismic surface effects. Here, we propose an alternative theoretical framework to address these issues. This novel framework makes use of Lagrangian PDF methods for turbulence modelling. In practice, these methods are implemented using random realisations of the flow, represented by a large number of notional particles tracked in a purely Lagrangian frame and evolved in time through stochastic differential equations. The Lagrangian nature of these methods proves naturally more suited to the description of stellar convection, which is an advection process by nature.

Keywords: Stellar physics, Turbulence, Convection

1 Introduction

Convection plays a crucial role in determining the structure and evolution of stars. In low-mass stars with a convective envelope, it modifies the seismic frequencies by altering the behaviour of acoustic waves in the surface layers (Christensen-Dalsgaard & Thompson 1997). For lack of a predictive model for these surface effects, this limits our ability to infer the mass, radius and age of these stars, or their internal structure, using asteroseismology.

Convection is usually included in stellar models using Mixing Length Theory (MLT), but it does not capture the properties of convective transport at interfaces (such as the bottom boundary of the convective envelope of low-mass stars, or their surface). In particular, it does not allow to predict surface effects (Belkacem & Samadi 2013; Samadi et al. 2015; Houdek & Dupret 2015), and it fails to account for convective penetration in neighbouring radiative zone, thus impacting age determination of intermediate stars with convective cores (Lebreton et al. 2014a,b). While alternative methods have been devised, based for instance on Reynolds-stress models (Xiong et al. 2000; Kupka et al. 2022; Ahlborn et al. 2022), and 3D hydrodynamic simulations have helped shed insight on the matter (Belkacem et al. 2019; Zhou et al. 2019; Schou & Birch 2020), the problem remains very much open to this day.

In this context, Probability Density Function (PDF) methods offer an invaluable modelling alternative (see Haworth 2010, for a review). Instead of a set of transport equations for means and second-order moments of the turbulent quantities (such as velocity or temperature), a single transport equation for their one-point, one-time PDF is solved. In practice, random realisations of the PDF are computed and evolved in time, using Lagrangian particle methods, where the flow is represented by a large set of notional particles whose trajectory and physical properties are tracked over time, and whose evolution is governed by a Lagrangian stochastic model

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of turbulence (e.g. Pope & Chen 1990; Welton & Pope 1997; Delarue & Pope 1997; Almeida & Navarro-Martinez 2021). This has several advantages compared to Large-Eddy Simulations (LES): advective transport is naturally included, and does not need to be explicitly modelled; turbulent dissipation can be modelled physically, while it is dominated by artificial dissipation due to finite grid resolution in LES; and actual ensemble averages are modelled, with dependence on time and space, while LES must rely on proxies like time averages or horizontal averages. We had previously used this class of models to derive semi-analytical prescriptions for the impact of convection on acoustic waves (Philidet et al. 2021, 2022).

Here, we present a new numerical code designed to model turbulent convection through such methods. We first present how we established a Lagrangian stochastic model adapted to stellar convection, after which we present how we implemented it in a new code.

2 Lagrangian stochastic description of stellar convection

The notional particles are described by their position $\mathbf{x}^*$, velocity $\mathbf{u}^*$, internal energy e^* and turbulent dissipation rate ϵ^* (or equivalently their turbulent frequency $\omega^* \equiv \epsilon^*/k$, where k is the turbulent kinetic energy). In their general form, these quantities evolve according to the following set of stochastic differential equations

$$\begin{aligned}
\mathrm{d}\mathbf{x}^* &= \mathbf{u}^*\mathrm{d}t \ , \\
\mathrm{d}\mathbf{u}^* &= \mathbf{a_u}(\mathbf{x}^*, t; \mathbf{u}^*, e^*, \omega^*) + b_u(\mathbf{x}^*, t; \mathbf{u}^*, e^*, \omega^*)\mathrm{d}\mathbf{W_u} \ , \\
\mathrm{d}e^* &= a_e(\mathbf{x}^*, t; \mathbf{u}^*, e^*, \omega^*) + b_e(\mathbf{x}^*, t; \mathbf{u}^*, e^*, \omega^*)\mathrm{d}W_e \ , \\
\mathrm{d}\omega^* &= a_\omega(\mathbf{x}^*, t; \mathbf{u}^*, e^*, \omega^*) + b_\omega(\mathbf{x}^*, t; \mathbf{u}^*, e^*, \omega^*)\mathrm{d}W_\omega \ ,
\end{aligned}$$

where each $\mathrm{d}W_X$ is the increment over $\mathrm{d}t$ of mutually independent Wiener processes, so that $\overline{W} = 0$ and $\overline{W(t)W(t+\mathrm{d}t)} = \delta(\mathrm{d}t)$, where δ is the Dirac distribution.

The difficulty lies in expressing the coefficients a_X in the deterministic part and the coefficients b_X in the stochastic (or random) part as a function not only of time and space, but also of the particle properties. This is done by deriving the transport equations for the mean flow (mean density $\overline{\rho}$, velocity $\widetilde{\mathbf{u}}$, internal energy $\widetilde{e}$, and turbulent dissipation $\widetilde{\omega}$, where we introduced the density-weighted average $\widetilde{X} \equiv \overline{\rho X}/\overline{\rho}$) and second-order moments (Reynolds stress tensor $\widetilde{u_i''u_j''}$, internal energy variance $\widetilde{e''^2}$, and internal energy flux $\widetilde{u_i''e''}$, where we introduced the fluctuations $X'' \equiv X - \widetilde{X}$), from the Lagrangian stochastic equations, and then comparing them to their exact counterpart deduced from first principles (that is from the equations of continuity, Navier-Stokes, and the principles of thermodynamics).

The resulting expressions for the coefficients a_X and b_X depend not only on the stochastic variables, but also on the mean flow properties. In order to express the mean flow quantities from the set of notional particles and their stochastic properties, we couple the Lagrangian stochastic model with a kernel estimation scheme for the ensemble averages, akin to Smoothed Particle Hydrodynamics (SPH). The idea is to average the adequate particle property locally, using a weighting kernel function $K(\mathbf{x})$ to filter particles in the vicinity of the location at which the mean is to be estimated. For example, the mean density would be given by

$$\overline{\rho}(\mathbf{x}, t) = \sum_{k=1}^{N} \Delta m K\left(\mathbf{x}^{*(k)}(t) - \mathbf{x}\right) \ , \tag{2.1}$$

where N is the total number of particles, and Δm is the mass assigned to each particle, and any density-weighted mean quantity can then be estimated thus

$$\widetilde{Q}(\mathbf{x}, t) = \frac{1}{\overline{\rho}} \sum_{k=1}^{N} \Delta m \ Q^{*(k)}(t) K\left(\mathbf{x}^{*(k)}(t) - \mathbf{x}\right) \ . \tag{2.2}$$

Together, the Lagrangian stochastic model for the evolution of particle properties, and the kernel estimation scheme for the means on which the stochastic model hinges, requires no spatial grid on which to discretise the equations, which makes its numerical implementation conceptually easy.

3 Convectively unstable simulation

To showcase the capabilities of this class of models, we carried out a two-dimensional implementation, where we only follow two components of particle positions, but all three components of their velocity, assuming that

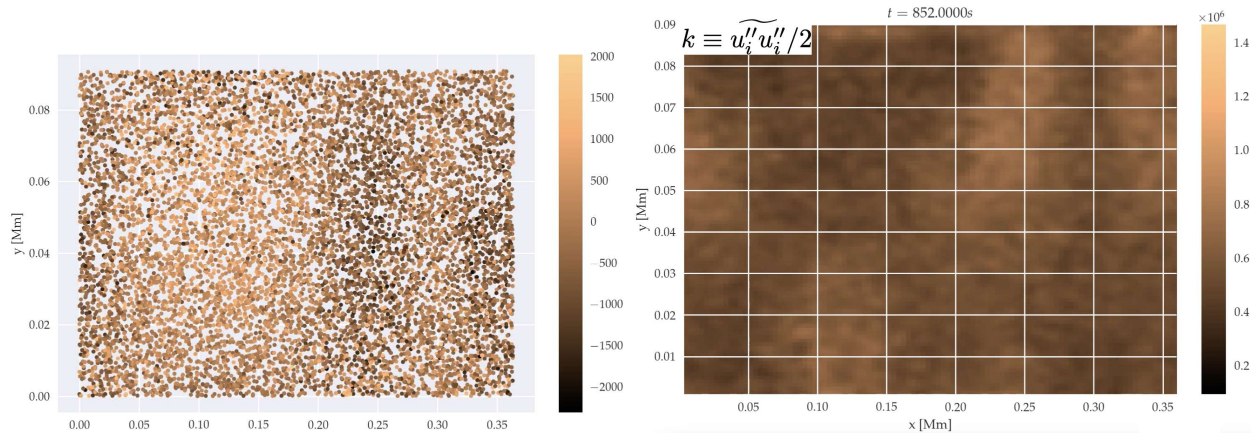

Fig. 1. Left: Snapshot of the particle vertical velocities in the simulation described in the text. Each dot represents one particle. Only 10,000 particles are shown for readability, whereas the simulation contains 500,000. **Right:** Snapshot of the turbulent kinetic energy, for the same time step.

the mean flow properties are uniform in the third direction. Periodic boundary conditions are enforced in the horizontal direction, and wall boundary conditions at the top and bottom of the domain. A linear mean internal energy profile is imposed in a small layer at the top, and another small layer at the bottom, controlling the degree of unstable stratification of the domain globally.

Fig. 1 showcases the output produced by the code. The left panel shows a snapshot of the primary output (in this case the vertical velocity of each particle), while the right panel shows the mean flow map (in this case the turbulent kinetic energy $k \equiv \widetilde{u_i''u_i''}/2$) for the same time step, extracted from the particle properties using the kernel estimation scheme. The simulation features distinct upflows and downflows characteristic of convectively unstable flows. The right panel, in particular, showcases the ability of the model to yield time- and space-dependent ensemble averages, in a way which traditional LES do not allow, one of the advantages of the present code.

The simulation also features oscillations, which are visible for example in the vertical velocity spectrum in the form of peaks at the resonant frequencies of the domain (see left panel of Fig. 2). Therefore, the issue of convection-oscillation coupling, and in particular that of seismic surface effects, may be investigated with our code.

The Lagrangian nature of the code also makes it much easier to incorporate Lagrangian trackers than it is in grid-based LES, thus allowing to investigate turbulent diffusion under varied physical conditions. The right panel of Fig. 2, for instance, shows the mean particle travel distance as a function of time increment, for different values of the imposed global internal energy gradient. For long time increments τ, the curves are consistent with a diffusive process characterised by an effective diffusion coefficients $\nu_{\mathrm{turb}} \sim 500$ km^2/s, in line with expected values from solar observations for instance (e.g. Rincon et al. 2025).

4 Conclusions

The new Lagrangian stochastic model presented here for the one-point statistics of stellar turbulent convection, and the new code designed to numerically implement this model, presents several advantages compared to more traditional LES: possibility to physically include the effect of turbulent dissipation, as well as the ability to extract time- and space-dependent maps of the different first- and second-order statistics of turbulence, offer the possibility to investigate surface effects in a way that is not possible to previously used 3D hydrodynamic simulations.

Furthermore, the Lagrangian nature of the code makes it much more convenient to track Lagrangian tracers, and therefore to measure turbulent diffusion. This therefore promises that this code will be helpful in investigating the efficiency of turbulent transport in overshooting regions, that is zones where convective plumes penetrate into neighbouring radiative zone. This problem, which remains open today, has an important impact on the evolution of stars harbouring such a convective/radiative interface, and as such on our capacity to scientifically exploit the data from the future PLATO mission.

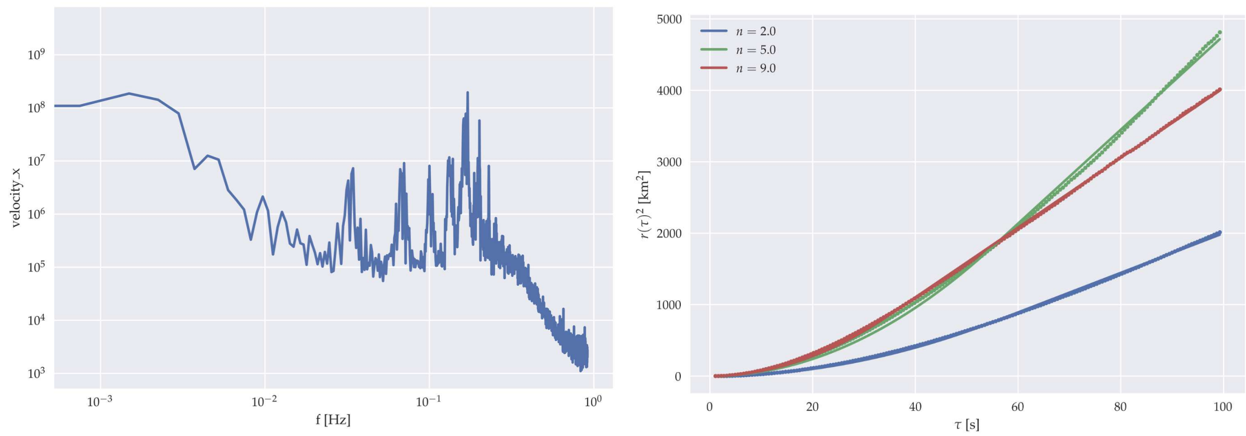

Fig. 2. Left: Spatial average of the vertical velocity power spectrum. Resonant frequencies of the domain appear as narrow peaks. **Right:** Squared particle dispersion distance r^2 as a function of time increment τ, averaged over all particles trajectories. The dots show the simulation data, and the solid line shows a fit to the data characterised by $r \propto \tau$ for small times, $r \propto \sqrt{\tau}$ for long times, and a continous transition between the two regimes. The different colours correspond to different values of n, related to the imposed global internal energy gradient through $n = (1 + (e_{\rm top} - e_{\rm bottom})(\Gamma_1 - 1)/g(x_{\rm top} - x_{\rm bottom}))^{-1}$.

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