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TRANSPORT BY THE TAYLER INSTABILITY: THE IMPORTANCE OF THE NUMERICAL IMPLEMENTATION

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Abstract. Traditionally, the transport of angular momentum in stellar radiative zones (RZ) is modelled as the result of angular momentum advection by meridional circulation and diffusion of angular velocity through shear-induced turbulence. Clearly, the angular velocity (Ω) profiles obtained with this model do not agree with the observations. These discrepancies argue for the consideration of additional transport mechanisms. Diffusion of angular velocity induced by the Taylor instability (TI) is one of the most promising candidates, as shown by previous studies. However, its implementation is not trivial, and it often leads to numerical errors that hinder a proper computation of rotation rates. We study the impact of different implementations of this process and advocate for the development of numerical methods able to derive reliable and reproducible outcomes.

1 Introduction

The wealth of asteroseismic data collected by space-borne missions CoRoT (e.g., Baglin et al. 2006) and *Kepler* (e.g., Borucki et al. 2010) for thousands of other stars, have raised thorny issues about the internal dynamics of low-mass stars. In particular, rotation is a key element of the stellar structure as it can affect convection, magnetic field, chemicals transport, its own geometry, and ultimately, fundamental parameters as the age of stars (e.g., Lebreton & Goupil 2014).

Most of the one-dimensional stellar evolution codes that can model the transport of angular momentum transport are based on the shellular rotation hypothesis proposed by Zahn (1992), which assumes that the differential rotation is predominantly radial owing to a very strong horizontal turbulence. In this picture, angular momentum is advected by the meridional circulation and angular velocity is diffused by shear-induced turbulence in radiative regions. Convective regions are supposed to be so turbulent that they rotate as solid body. This formalism of the internal angular momentum transport (e.g. Aerts et al. 2019) does not reproduce observations: the core rotation rate in stars is much lower than predicted (e.g., Marques et al. 2013; Deheuvels et al. 2014; Cantiello et al. 2014). This advocates to include additional transport mechanisms in radiative zones. Many have been suggested and they fall into two categories: extraction of angular momentum by waves (e.g., Talon & Charbonnel 2003; Belkacem et al. 2015; Pinçon et al. 2016), or diffusion of angular velocities by hydro- or magneto-hydrodynamic (MHD) instabilities (e.g. Palacios et al. 2003; Maeder & Meynet 2004; Caleo et al. 2016).

Numerous work studied the transport by the Tayler instability (TI) mechanisms (Spruit 2002; Maeder & Meynet 2004; Fuller et al. 2019; Petitdemange et al. 2023). The transport induced by this instability is included in stellar evolution codes using a simple diffusive approach associated with an effective viscosity deduced through heuristic arguments. Different prescriptions exist, depending on different parametrizations and lead to different conclusions on the efficiency of this process along stellar evolution (Eggenberger et al. 2005; Cantiello et al. 2014; Wheeler et al. 2015; Eggenberger et al. 2019). In general, all these prescriptions allow to reproduce observations qualitatively at a given evolutionary stage only.

On our path to implementing and studying the different available prescriptions with the stellar evolution code Cesam2k20 (Morel 1997; Morel & Lebreton 2008; Marques et al. 2013, Manchon et al., subm.), we stumble

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over a numerical issue inherent to the resolution of transport equation involving highly diffusive processes with finite volume methods. The characteristic time of the diffusion associated to the TI ($\sim$ 100 years) is very small compare to the time step used for stellar evolution computation, of the order of a few hundredth of the Kelvin-Helmholtz (0.1 Myr) or nuclear timescales (10 Myr) depending on the phase of the evolution. It results in the formation of stairs in the angular velocity profile, with a purely numerical origin.

In this work, we explore the dependence of the predicted evolution of the angular velocity profile to various choices of numerical implementation. Section 2 recalls the formalism developed by Maeder & Meynet (2004) to describe the TI. We restricted to this prescription throughout the paper. Section 3 illustrate the effect of different numerical control parameters on the rotation rates. We conclude in Sect. 4

2 Maeder & Meynet (2004)'s formalism of the Tayler instability in a nutshell

Considering an angular velocity profile Ω in the radiative zone with a differential rotation $q = |\partial \ln \Omega / \partial \ln r|$ Maeder & Meynet (2004) derive a criteria for the onset of the TI, which is

$$q > \left(\frac{N}{\Omega}\right)^{7/4} \left(\frac{\eta}{r^2 N}\right)^{1/4} = q_{\rm min}, \tag{2.1}$$

which induces a turbulent viscosity

$$\nu_{\rm TI} = \frac{r^2}{q\Omega}\frac{\omega_{\rm A}^3}{N} = \frac{\Omega r^2}{q}\left(\frac{\omega_{\rm A}}{\Omega}\right)^3 \frac{\Omega}{N}, \tag{2.2}$$

where N is the Brunt-Väissälä frequency, N_T and N_μ its thermal and molecular components, $\omega_{\rm A}$ is the Alfvèn frequency and η a turbulent magnetic diffusivity induced by the TI.

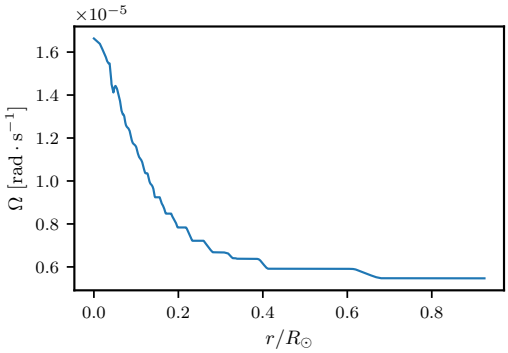

Fig. 1. Angular velocity profile as a function of the normalized radius for a $1M_\odot$ model, at an age of 1400 Myr. The TI is described according to Maeder & Meynet (2004).

3 Impact of the implementation on the angular velocity

Cesam2k20 solves the advecto-diffusive equation for the transport of angular momentum according to Zahn (1992):

$$\rho \frac{{\rm d}r^2\Omega}{{\rm d}t} = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\rho r^4 \Omega U_2\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho \nu_{\rm v} r^4 \frac{\partial \Omega}{\partial r}\right), \tag{3.1}$$

where U_2 is the vertical velocity of the meridional circulation, and $\nu_{\rm v}$ is a viscosity that, in this case, sums the effect of the TI and of the shear-induced turbulence according to Palacios et al. (2003). This equation is solved with a finite-volume method and a relaxation scheme (Press et al. 1995).

Figure 1 shows the profile of Ω in a $1M_\odot$ model at 1.4Gyr. We distinctly see multiple stairs in the profile. This behaviour is also found in models computed by Maeder & Meynet (2004); Cantiello et al. (2014); Wheeler et al. (2015); Fuller et al. (2019). They have a simple numerical origin, illustrated in Fig. 2. Cesam2k20 decouples odd and even time steps in the sense that the viscosity of the actual time step is computed with angular velocity computed at previous time step. We start our picture at time t_0 in a radiative zone with a

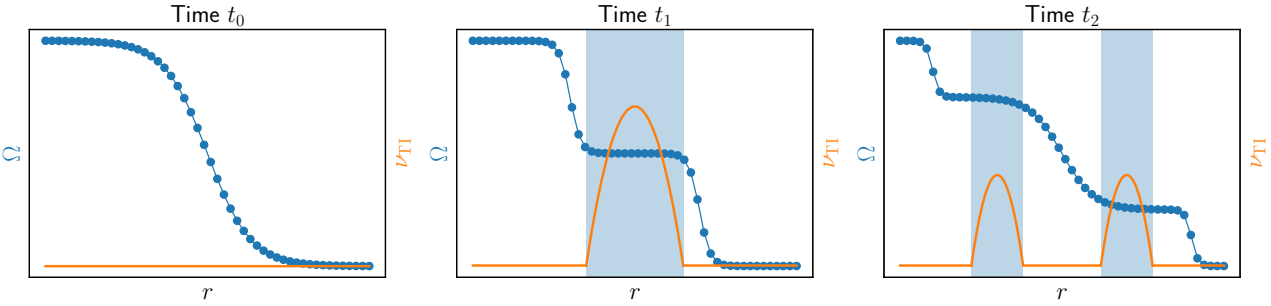

Fig. 2. Sketch representation of the formation of stairs in the profile of angular velocity. Blue lines represent profiles of Ω in arbitrary units at three successive time step of a stellar evolution. Blue shaded regions are regions where the Tayler instability could develop at previous time step. Orange lines represent the value of the induced turbulent viscosity, in arbitrary units.

strong shear in the middle (Fig. 2, left panel) and a null viscosity. At next time t_1 (Fig. 2, middle panel), a high viscosity is induced in the middle of the zone where the shear was very strong at t_0, which smoothes completely Ω in the region where the Tayler instability could grow. This cause the formation of three regions where Ω is almost flat, linked by two transition zones of intense shear. At the next time step t_2 (Fig. 2, right panel), the viscosity computed with $\Omega(t_1)$ is null everywhere, except in the two unstable regions of high shear. This causes the appearance of two new stairs, and an additional sheared region. And so a stair keeps building, time step after time step. In a grid mesh uniformly spaced with a step Δx, the typical time of diffusion is given by $t_{\rm diff} = \Delta x^2/\nu_{\rm TI}$. In the case of the Tayler instability, it can typically reach 10^7 cm$^2 \cdot$s^{-1} and with 1000 layers per solar radius, one reaches a time of 100 years, much smaller than the usual time step of a stellar evolution, which is of the order of 10 Myr during the main sequence.

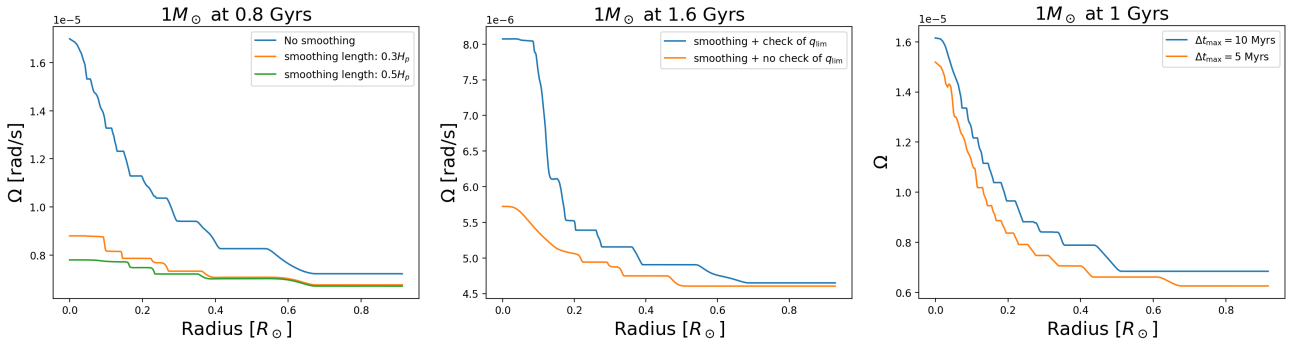

Fig. 3. Rotating $1M_\odot$ models computed with same initial rotation conditions, and different numerical control parameters for the TI. Angular velocity profiles are displayed at different evolutionary phase in each panel. *Left*: Impact of various smoothing lengths: 0.0 (blue), $0.3H_p$ (orange), $0.5H_p$ (green). *Middle*: TI is assumed to develop only when $q > q_{\rm lim}$ (blue) or everywhere (orange). *Right*: model computed with a time step limited to 10 Myr (blue), or 5 Myr (orange).

In order to solve this issue, some team make use of a smoothing scheme applied on various quantities such as $\nu_{\rm TI}$, or Ω (see, e.g. Fuller et al. 2019, who smooth $\nu_{\rm TI}$ over a fixed number of layers). We designed our own scheme in order to smooth $\nu_{\rm TI}$ over a kernel of a given length. The characteristic scale of the instability derived by Spruit (2002) with a linear analysis is very small and probably not realistic. We therefore tried to compute rotating models where $\nu_{\rm TI}$ is smoothed over various fractions of H_p (see. Fig. 3, left panel). This comparison clearly demonstrates a strong dependence of the final rotation rates on the chosen length.

If the typical length of the instability represent a substantial fraction of the radiative zone extent, and if one assumes that, once generated, the magnetic field can survived for a much longer time than $t_{\rm diff}$ computed above, one could argue that the fluid keeps a viscosity $\nu_{\rm TI}$, even when the criteria $q > q_{\rm lim}$ is not valid. Results obtained with this test are plotted in Fig. 3, middle panel. Not checking the criteria (2.1) makes the instability more efficient at diffusing angular velocity and lead again to very different Ω-profiles.

Finally, we assessed the impact on the time step Δt chosen to compute the stellar evolution. Cesam2k20

adapts the timestep to the phase of the stellar evolution, but we limited this value to 10 or 5 Myr (see. Fig. 3, right panel). The fact that we obtain different Ω means that the value of Δt is not adapted to the proper computation of the effect of the TI.

4 Conclusions

In this work we implemented in the stellar evolution code Cesam2k20 various numerical treatment of the Tayler instability following the formalism of Maeder & Meynet (2004). A simple implementation reveals the creation stairs in the computed angular velocity profiles. These findings are compatible to what is generally found in the literature and is purely numerical. The origin of these stair patterns is linked to the decoupling of the diffusion induced by the TI and Ω between odd and even time steps, and to the very short timescale on which the TI acts, compare to the typical time step used in stellar evolution code.

We assessed the impact of several implementations and numerical control parameters and exhibit very different rotation rates and very different stair patterns. Such a large dependence on the numerical parameters may suggest that the results exhibited in the literature might be qualitatively correct, but not quantitatively.

Our goal is now to design a proper implementation of the TI for stellar evolution computations. Several ways could be investigated such as constraining a physical length for the smoothing of TI-induced viscosities or designed numerical methods making the computation of stellar evolution with time steps of the order of 100 yrs conceivable. It will be the topic of a forthcoming paper.

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