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NONLINEAR DIFFUSION EQUATION FOR ALFVÉN WAVE TURBULENCE

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Abstract. We discuss about the possibility to derive rigorously a nonlinear diffusion equation for incompressible MHD turbulence. The background of the analysis is the asymptotic Alfvén wave turbulence equations from which a differential limit is taken. The result is a universal diffusion-type equation in **k**-space which describes in a simple way and without free parameter the energy transport perpendicular to the external magnetic field $\mathbf{B_0}$ for transverse fluctuations. It is compatible with both the thermodynamic equilibrium and the finite flux spectra derived by Galtier et al. (2000). This new system offers a powerful description of a wide class of astrophysical plasmas.

Keywords: magnetohydrodynamics, turbulence, waves

The measurements made in particular by various spacecrafts have added substantially to our knowledge of astrophysical plasmas. Among the different media widely analyzed we find the interstellar medium, the solar wind or the Sun's atmosphere. In the latter case, it is believed that magnetohydrodynamic (MHD) turbulence plays a central role in the dynamics and the small scale heating. For example, in active region loops spectrometer analyses revealed non-thermal velocities reaching sometimes 50 km/s (Chae et al. 1998); this line broadening is generally interpreted as unresolved turbulent motions with length scales smaller than the diameter of coronal loops which is about one arcsec and timescales shorter than the exposure time of the order of few seconds. Turbulence is evoked in the solar coronal heating problem since it offers a natural process to produce small scale heating (see *e.g.* Heyvaerts & Priest 1992). Weak MHD turbulence is now proposed has a possible regime for some coronal loops since a very small ratio is expected between the fluctuating magnetic field and the axial component (Rappazzo et al. 2007). Inspired by the observations and by recent direct numerical simulations of 3D MHD turbulence (Bigot et al. 2008a), an analytical model of coronal structures has been proposed (Bigot et al. 2008b) where the heating is seen as the end product of a wave turbulent cascade. Surprisingly, the heating rate found is non negligible and may explain the observations.

MHD turbulence modeling is the main tool to investigate the situations previously discussed. Although it cannot be denied that numerical resources have been significantly improved during the last decades, direct numerical simulations of MHD equations are still limited for describing highly turbulent media (see *e.g.* Mininni & Pouquet 2007). For that reason, shell cascade models are currently often used to investigate the small scale coronal heating (Buchlin & Velli 2007) and its impact in terms of spectroscopic emission lines. Transport equations are also used for example in the context of solar wind acceleration in the extended solar corona (Cranmer & van Ballegooijen 2003). The *ad hoc* model is an advection-diffusion equation for the evolution of the energy spectrum whose inspiration is found in the original paper by Leith (1967). It is also a cascade model where the locality of the nonlinear interactions is assumed but where the dynamics is given by a second-order nonlinear partial differential equation whereas we have ordinary differential equations for shell models.

A theoretical understanding of the statistics of turbulence and the origin of the power law energy spectrum, generally postulated from dimensional considerations *à la* Kolmogorov, remains one of the outstanding problems in classical physics which continues to resist modern efforts at solution. The difficulty lies in the strong nonlinearity of the governing equations which leads to an unclosed hierarchy of equations. Faced with that situation different models have been developed like closure models in Fourier space. In the meantime, Leith (1967) and Iroshnikov (1964) proposed a diffusion approximation to inertial energy transfer for respectively isotropic HD and MHD turbulence. This new class of *ad-hoc* models describes the time evolution of the spectral energy density for originally an isotropic 3D incompressible turbulence, in terms of a partial differential equation

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by making a diffusion approximation to the energy transport process in the **k**–space representation. Ignoring forcing and dissipation the Leith's equation for HD turbulence reads in Fourier space

$$\frac{\partial E(k)}{\partial t} = \frac{\partial}{\partial k}\left(k^{11/2}\sqrt{E(k)}\frac{\partial}{\partial k}\left(\frac{E(k)}{k^2}\right)\right), \tag{0.1}$$

where $E(k)$ is the unidimensional energy spectrum. Beyond its relative simplicity, equation (0.1) exhibits several important properties like the preservation after time integration of a non negative spectral energy and the production of the Kolmogorov spectrum, $k^{-5/3}$, in the inertial range.

The generalization of the Leith's model to 3D isotropic MHD turbulence was first made by Iroshnikov (1964) and then in a different way by Zhou & Matthaeus (1995). In the latter case, the main modification happens in the evaluation of the transfer time for which a combination of the eddy turnover time τ_{eddy} and the Alfvén time τ_A was proposed. The phenomenological evaluation of the transfer time allows the recovering of either the Kolmogorov, $k^{-5/3}$, or the Iroshnikov–Kraichnan, $k^{-3/2}$, spectrum when the ratio τ_{eddy}/τ_A is respectively much less or much larger than one.

The case of Alfvén wave turbulence for which a relatively strong B_0 is required is an important limit for which a rigorous analysis is possible (Galtier et al. 2000). The wave kinetic equations derived are a set of coupled integro-differential equations which are not obvious to handle numerically or analytically. This remark was a motivation for deriving a simpler set of equations by taking the differential limit of the asymptotically exact Alfvén wave equations. The systematic derivation gives in the simplest case (*i.e.* for zero cross-helicity) the following equation for the transverse fluctuations (Galtier & Buchlin 2010)

$$\frac{\partial E_\perp(k_\perp)}{\partial t} = \frac{\partial}{\partial k_\perp}\left(k_\perp^6 E_\perp(k_\perp)\frac{\partial}{\partial k_\perp}\left(\frac{E_\perp(k_\perp)}{k_\perp}\right)\right), \tag{0.2}$$

where $E_\perp(k_\perp)$ is the energy spectrum and $k_\perp$ is the wavenumber transverse to the uniform magnetic field $\mathbf{B_0}$. This nonlinear diffusion equation reproduces the finite flux solution in $k_\perp^{-2}$ as well as the thermodynamic equilibrium spectrum in $k_\perp$ (see Galtier & Buchlin (2010) for numerical illustrations). It is a simple and therefore useful system for describing a wide class of astrophysical plasmas. Coronal magnetic loops characterized by a strong axial magnetic field are probably a first good example of application of Alfvén wave turbulence.

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