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CRITICAL LAYERS FOR INTERNAL WAVES IN STELLAR RADIATION ZONES

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Abstract. Internal waves are candidates for explaining stellar radiation zone angular velocity profiles, for example in the Sun. The equation describing the adiabatic and inviscid propagation of such waves is singular at a “ critical level” where the fluid rotation frequency is proportional to the wave frequency. Here, we propose a generalization of previous studies to the stellar, spherical case. We resolve the problem by three methods, depending on the value of the Richardson number at the critical level Ri_c and our results can be divided into two cases. If $(l(l+1)/m^2)\ Ri_c$ is larger than 1/4, waves are attenuated by a factor depending on Ri_c as they pass through a critical level. If $(l(l+1)/m^2)\ Ri_c$ is less than 1/4, the basic flow is then unstable and the wave can be reflected with a coefficient larger than 1. In the latter case, the process is called over-reflection.

Keywords: internal gravity waves, critical layers, angular momentum transport

1 Introduction

Internal waves have been invoked as a source of mixing for chemicals (Press 1981; Schatzman 1996), before being used to explain the evolution of angular velocity (Goldreich & Nicholson 1989; Charbonnel & Talon 2005) due to the transport of angular momentum they induce. Later, Barker & Ogilvie (2010) studied internal waves approaching the center of a solar type star. The behaviour of such waves at a critical level has mainly been studied in Geophysics with 2D cartesian coordinates (Booker & Bretherton 1967; Lindzen 1985; Ringot 1998). This work is a generalization to the stellar, spherical case.

2 Waves in differentially rotating stars

2.1 Hypotheses

As a first approach we consider a perfect fluid (no viscosity and heat conduction). We neglect the Coriolis and Lorentz forces too and assume that the angular velocity Ω depends only on depth : $\Omega(r,\theta) = \Omega(r)$, justified by the fact that differential rotation in latitude is severely limited through hydrodynamical instabilities (Zahn 1992).
Then, we define the local frequency $\sigma(r) = \sigma_c + m\Delta\Omega(r)$ caused by the Doppler shift due to the differential rotation. σ_c is the excitation frequency of the wave and $\Delta\Omega(r) = \Omega(r) - \Omega_c$ is the difference between the local angular velocity and the one at the base of the convective zone or the top of the convective core, where internal gravity waves are excited. Waves are prograde if $m < 0$ and retrograde if $m > 0$ (m corresponds to a Fourier expansion along the longitudinal direction).

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2.2 System of linearized equations

We solve the adiabatic and inviscid system composed by :

$$\left\{\begin{array}{ll}\text{Equation of motion} & D_t\vec{u} = -\frac{\vec{\nabla}p'}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}}\vec{g},\\ \text{Continuity equation} & D_t\rho' + \vec{\nabla}.(\bar{\rho}\vec{u}) = 0,\\ \text{Energy equation} & D_t(\frac{\rho'}{\bar{\rho}} - \frac{1}{\Gamma_1}\frac{p'}{\bar{p}}) + (\frac{d\ln\bar{\rho}}{dr} - \frac{1}{\Gamma_1}\frac{d\ln\bar{p}}{dr})u_r = 0,\end{array}\right. \tag{2.1}$$

where $\vec{u}$ is the fluctuation of velocity associated with the wave, p and ρ the pressure and the fluid density, g the acceleration due to gravity, $\Gamma_1 = \frac{\partial\ln\bar{p}}{\partial\ln\bar{\rho}}$ the adiabatic exponent and $D_t = \partial_t + \Omega\partial_\varphi$. Mean values are denoted by $\bar{X}$ and the associated fluctuations by X'.
The stratification of the studied radiation zone is described with the Brunt-Väisälä frequency : $N^2 = -g(\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial r} - \frac{1}{\Gamma_1\bar{p}}\frac{\partial\bar{p}}{\partial r})$ where we neglect the perturbation of the gravitational field (Cowling approximation). We introduce the wave's Lagrangian displacement $\vec{\xi} = \xi_v\vec{e_r} + \vec{\xi}_h$ and following the method proposed by Zahn et al. (1997), we define $\xi_v = \hat{\xi}_{v;l,m}(r)Y_{l,m}(\theta,\varphi)e^{i\sigma_c t}$ ($Y_{l,m}$ are the usual spherical harmonics) and $\Psi_{l,m}(r) = \bar{\rho}^{1/2}r^2\hat{\xi}_{v;l,m}(r)$ to obtain a differential equation of second order for the vertical velocity :

$$\frac{d^2\Psi_{l,m}}{dr^2} + k_v^2(r)\Psi_{l,m} = 0, \tag{2.2}$$

where k_v is the vertical wavenumber whose expression near the critical layer is :

$$k_v^2 = (\frac{N^2}{\sigma^2} - 1)\frac{l(l+1)}{r^2} \underset{r\to r_c}{\approx} \frac{l(l+1)}{m^2}\frac{Ri_c}{(r-r_c)^2}. \tag{2.3}$$

In cartesian 2D coordinates the equivalent of this equation is called TGS (Taylor, Goldstein and Synge) and we will use this name again. From this point, we use the expression of k_v near the critical layer defined by $r = r_c$ where $\sigma(r_c) = \sigma_c + m\Delta\Omega(r_c) = 0$. The TGS equation is singular at this point.

3 Resolution of the TGS equation

The original contribution of this work is related to the resolution of the TGS equation in the spherical case. Most of the time people use the WKBJ approximation. In spherical coordinates, the Richardson number is $Ri = \frac{N^2}{r^2\Omega'^2(r)}$ and we denote it Ri_c when we evaluate it at the critical level $r = r_c$. After some calculation it appears that the WKBJ method is available only in the case where $\frac{l(l+1)}{m^2}Ri_c >> \frac{1}{4}$, that is to say if the shear is weak before the stratification.

3.1 Case 1 : $\frac{l(l+1)}{m^2}Ri_c \geq \frac{1}{4}$

Another method is available in the whole domain $\frac{l(l+1)}{m^2}Ri_c \geq \frac{1}{4}$. It is called the Frobenius method and the solution are the same as those obtained with the WKBJ method. Nevertheless we have to cut the physical domain into two parts to avoid the singularity in $r = r_c$. Then, it is possible to connect the solutions with the method employed by Booker & Bretherton (1967). The global solution is of the form :
Above the critical layer ($r > r_c$) :

$$\Psi_{l,m+} = C|r-r_c|^{\frac{1}{2}\left(1+i\sqrt{q_{l,m}-\frac{1}{4}}\right)} + D|r-r_c|^{\left(1-i\sqrt{q_{l,m}-\frac{1}{4}}\right)}, \tag{3.1}$$

and below the critical layer ($r < r_c$) :

$$\Psi_{l,m-} = -iC|r-r_c|^{1+i\nu_{l,m}}e^{-\pi\nu_{l,m}} - iD|r-r_c|^{1-i\nu_{l,m}}e^{+\pi\nu_{l,m}}, \tag{3.2}$$

where $\nu_{l,m} = \sqrt{\frac{l(l+1)}{m^2}Ri_c - \frac{1}{4}}$.

We observe two phenomenon as the wave passes through the critical level :

- a **phase difference** of $-\frac{\pi}{2}$,
- an **attenuation** by a factor $e^{-\pi\sqrt{\frac{l(l+1)}{m^2}Ri_c - \frac{1}{4}}}$ which is equivalent to the factor found by Booker & Bretherton (1967) in cartesian coordinates. This factor is represented in the figure 1.

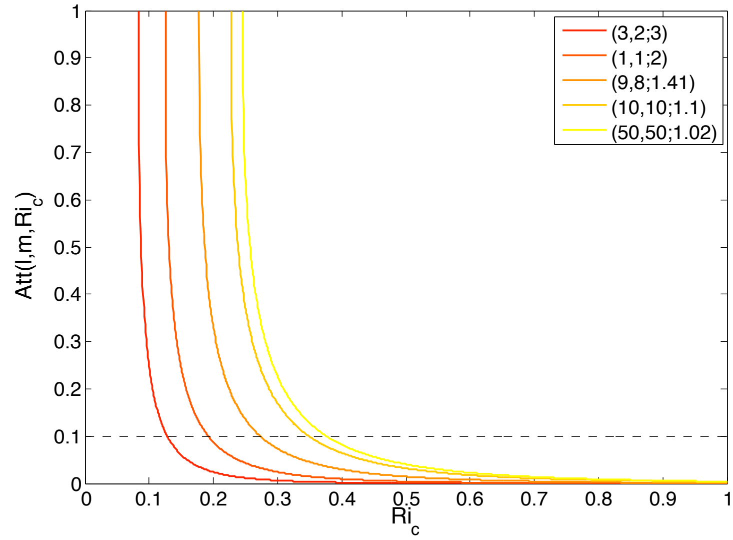

Fig. 1. Attenuation's rate Att=$e^{-\pi\sqrt{\frac{l(l+1)}{m^2}Ri_c-\frac{1}{4}}}$ for different values (l,m) as a fonction of Ri_c. For each plot, the minimal value of Ri_c is $\frac{m^2}{4l(l+1)}$. Legend : (l,m;$\frac{l(l+1)}{m^2}$).

3.2 *Case 2 :* $\frac{l(l+1)}{m^2}Ri_c < \frac{1}{4}$

In this case, the Frobenius method is available but not physically acceptable. That the reason why we introduce new variables : $\mu_{l,m} = \sqrt{\frac{1}{4} - \frac{l(l+1)}{m^2}Ri_c}$ and $X = \frac{l(l+1)}{r_c^2}(r - r_c)$.
The TGS equation becomes :

$$\frac{d^2\Psi_{l,m}}{dX^2} + \left(\frac{\frac{1}{4} - \mu_{l,m}^2}{X^2} - 1\right)\Psi_{l,m} = 0, \tag{3.3}$$

and the solutions are combinations of **modified Bessel functions** :

$$\Psi_{l,m-}(X) = X^{\frac{1}{2}}\left[K_1 I_{\mu_{l,m}}(X) + K_2 I_{-\mu_{l,m}}(X)\right]. \tag{3.4}$$

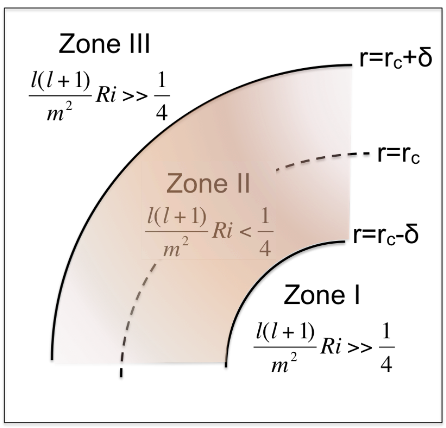

Fig. 2. Expected profile of the area around the critical layer.

Lindzen & Barker (1985) have studied this case in 2D cartesian coordinates. We employ the same method to calculate the reflection and transmission coefficients of the wave passing through the critical layer. Let assume that the area around the critical layer has the profile described on the figure 2. A wave created in the first zone can be described by the system above.

$$\begin{cases} \text{zone I :} & \Psi_1(r) = & e^{-ik_v r} + Re^{ik_v r}, \\ \text{zone II :} & \Psi_2(r) = & (r - r_c)^{1/2}\left[K_1 I_\mu\left(k_h(r-r_c)\right) + K_2 I_{-\mu}\left(k_h(r-r_c)\right)\right], \\ \text{zone III :} & \Psi_3(r) = & Te^{ik_v r}. \end{cases}$$

The constants K_1, K_2, R and T are determined by the requirements that Ψ and $\frac{d\Psi}{dr}$ should be continuous at $r = \pm\delta$ (continuity of displacement and stresses). We obtain the figure 3 which show that $|R|$ and $|T|$ are greater than one if Ri_c is small enough. These graphs are similar to those obtained in the cartesian case by Lindzen (1985). In this case, waves are taking supplementary energy from the unstable shear. Let us point out that the expressions of R and T are independent of the sign of m.

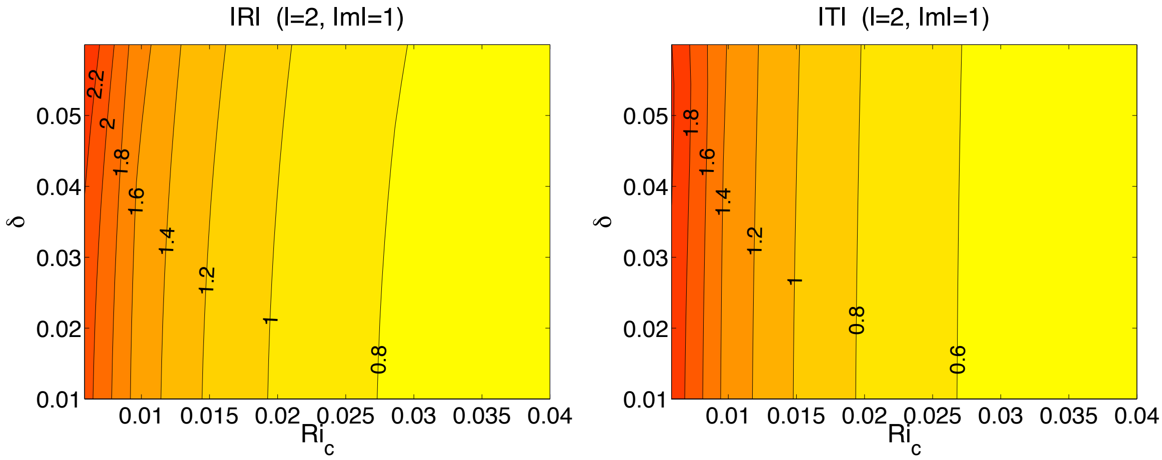

Fig. 3. Left: Reflection coefficient as a function of Richardson number Ri_c and width δ of the region where $\frac{l(l+1)}{m^2}Ri_c < \frac{1}{4}$, for l=2 and $|m|$=1. **Right:** Transmission coefficient for the same example.

4 Conclusion

If the Richardson number at the critical layer satisfies $\frac{l(l+1)}{m^2}Ri_c > \frac{1}{4}$, the corresponding internal gravity wave passing through the layer is attenuated by the factor $\exp(-\pi\sqrt{\frac{l(l+1)}{m^2}Ri_c - \frac{1}{4}})$. But in the case where $\frac{l(l+1)}{m^2}Ri_c < \frac{1}{4}$, an over-reflection occurs. Extension of the previous analysis to a non perfect fluid show that the absorption mechanism is not dependent on viscosity or other dissipative processes. Moreover, work is in progress to implement these theoretical results in the dynamical stellar evolution code STAREVOL (Siess et al. 2000; Palacios et al. 2003; Decressin et al. 2009).

References

Barker, A. J. & Ogilvie, G. I. 2010, MNRAS, 404, 1849

Booker, J. & Bretherton, F. 1967, J. Fluid Mech., 27, 513

Charbonnel, C. & Talon, S. 2005, Science, 309, 2189

Decressin, T., Mathis, S., Palacios, A., et al. 2009, A&A, 495, 271

Goldreich, P. & Nicholson, P. D. 1989, ApJ, 342, 1079

Lindzen, R. Barker, A. 1985, J. Fluid Mech., 151, 189

Palacios, A., Talon, S., Charbonnel, C., & Forestini, M. 2003, A&A, 399, 603

Press, W. 1981, ApJ, 245, 286

Ringot, O. 1998, PhD thesis, Université Paris VII

Schatzman, E. 1996, Sol. Phys., 169, 245

Siess, L., Dufour, E., & Forestini, M. 2000, A&A, 358, 593

Zahn, J.-P. 1992, A&A, 265, 115
Zahn, J.-P., Talon, S., & Matias, J. 1997, A&A, 322, 320