SF2A 2011
G. Alecian, K. Belkacem, R. Samadi and D. Valls-Gabaud (eds)

ANALYTICAL EXPRESSION OF THE POTENTIAL GENERATED BY A MASSIVE INHOMOGENEOUS STRAIGHT SEGMENT

N.-E. Najid[1] and E. Elourabi[2]

Abstract. Potential calculation is an important task to study dynamical behavior of test particles around celestial bodies. Gravitational potential of irregular bodies is of great importance since the discoveries of binary asteroids, this opened a new field of research. A simple model to describe the motion of a test particle, in that case, is to consider a finite homogeneous straight segment. In our work, we take this model by adding an inhomogeneous distribution of mass. To be consistent with the geometrical shape of the asteroid, we explore a parabolic profile of the density. We establish the closet analytical form of the potential generated by this inhomogeneous massive straight segment. The study of the dynamical behavior is fulfilled by the use of Lagrangian formulation, which allowed us to give some two and three dimensional orbits.

Keywords: Potential, Inhomogeneous-distribution, Asteroids.

1 Introduction

The discovery of irregular small bodies and binary asteroids like Ida and Doctyl, gave a rise to the potential calculation. Many attempts have been made to approximate the potential. Riaguas et al. (1999) proposed a homogeneous straight segment. In Elipe et al. (2003) described the motions around (433) Eros with the same homogeneous model. A harmonic polyhedron was used by Werner and Scheeres for asteroid 4769 Castalia in Werner et al. (1997) and Werner. (1994). Ellipsoids, material points and double material segment was used by Przemyslaw et al. (2003)) and Przemyslaw et al. (2006), as the model of irregular elongated bodies. In our work we give a new idea to models the potential generated by an elongated body. We consider a straight massive segment with variable density. To be consistent with the geometrical aspect of the asteroid, we use a parabolic profile. Our work generalize that of Riaguas et al. (1999). In the first part of this work, we establish the closet forme of the potential generated by an inhomogeneous massive straight segment. In the second part we study the dynamical behavior of a test particle in the field of the straight segment. We conclude in the last part by the numerical resolution of the differential equations of motion. In this part we show some orbits in two and three dimension.

2 Potential calculation

We consider an inhomogeneous straight segment of length $2l$ and mass M which lies along the $(x - axis)$, with a parabolic profile of linear mass density expressed by $\lambda(x) = -ax^2 + b$ in which a and b are linked by $a < \frac{b}{l^2}$ and $M = -\frac{2}{3}al^3 + 2bl$. At a point P, the gravitational potential generated by the segment is :

$$U(P) = -G \int \frac{dm}{r} \tag{2.1}$$

Where G is the gravitational constant. r is the distance between P and the infinitesimal mass dm located at H with abscissa x_H in the segment. Fig.1.
Let us consider an inertial reference frame $(Oxyz)$, and let $\mathbf{r}_1$, and $\mathbf{r}_2$ be the position vectors of the end points

[1] Equipe Mécanique Statistique et Astronomie - Laboratoire de Physique Théorique et Appliquée Université Hassan II Ain Chock- Faculté des Sciences Ain Chock BP 5366 Maârif Casablanca Morocco.
[2] Equipe Mécanique Statistique et Astronomie - Laboratoire de Physique Théorique et Appliquée Université Hassan II Ain Chock- Faculté des Sciences Ain Chock BP 5366 Maârif Casablanca Morocco.

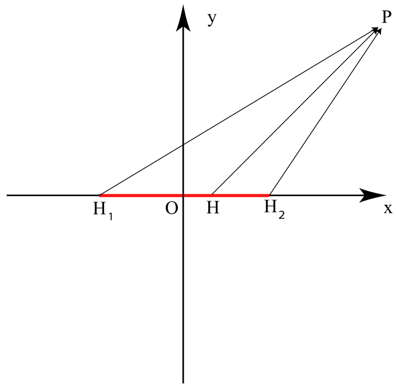

Fig. 1. Straight segment .

H_1 and H_2 of the straight segment. We define $\nu = \frac{1}{2}(1 + \frac{x_H}{l})$ as a new variable of integration with $0 \le \nu \le 1$. After calculation we obtain

$$U(r_1, r_2) = 4al^2 G \int_0^1 \frac{\nu^2 - \nu - \frac{b - al^2}{4al^2}}{\sqrt{\nu^2 + \nu \left(\frac{r_2^2 - r_1^2 - 4l^2}{4l^2}\right) + \frac{r_1^2}{4l^2}}} d\nu \tag{2.2}$$

After a laborious calculation and simplification we achieve the closet expression of the potential generated at P:

$$U(r_1, r_2) = \frac{G}{32l^2}\left(A_1 + A_2 \ln\left(\frac{r_2 + r_1 - 2l}{r_2 + r_1 + 2l}\right)\right) \tag{2.3}$$

Where

$$A_1 = 16al^3(r_2 + r_1) + 12al(r_2 - r_1)\left(r_1^2 - r_2^2\right)$$

and

$$A_2 = 8al^2(r_2 + r_1)^2 - 16al^2 r_1 r_2 - 3a(r_1 - r_2)^2(r_2 + r_1)^2 - 16al^4 + 32bl^2$$

We define $s = r_2 + r_1$, $d = r_1 - r_2$ and $p = r_2 r_1$ as an auxiliaries functions depending only on distances r_1 and r_2 of the particle to the end points of the segment. The expression of Eq. (2.3) reduce to :

$$U(P) = -\frac{G}{32l^2}\left\{12alsd^2 - 16al^3 s + \left[8l^2 a\left(s^2 - 2p\right) - 3as^2 d^2 - 16l^4 a + 32bl^2\right]\ln\left(\frac{s + 2l}{s - 2l}\right)\right\} \tag{2.4}$$

Eq. (2.4) represent the gravitational potential generated by an inhomogeneous straight segment with a quadratic profile of density, this expression is our main result, to have more details and study about, (see Najid et al. 2011). The case of constant density (Riaguas et al. 1999) is a particular situation of Eq. (2.4), if we put $a = 0$ and $b = \frac{M}{2l} = \lambda$. We obtain the expression (1) in Riaguas et al. (1999).

3 Dynamical study

We plane to study the dynamical behavior of a test particle, with unit mass, located at P in the field of the inhomogeneous straight segment. $R(O, x, y, z)$ is the sidereal referential frame, with the cylindrical coordinates $(\rho,\ \theta,\ x)$ as in Fig. 2. The differential equations of motion corresponding to ρ, x and θ are given by

$$\ddot{\rho} = \rho\dot{\theta}^2 + \frac{G}{32l^2 p}\left(A_3 - \frac{4l\rho s}{s^2 - 4l^2}A_4\right) \tag{3.1}$$

$$\ddot{x} = \frac{Ga}{16l^2 p}\left(A_5 + A_6 \ln\left(\frac{s + 2l}{s - 2l}\right) - \frac{2l\,(xs - ld)}{s^2 - 4l^2}A_7\right) \tag{3.2}$$

$$\rho^2\dot{\theta} = \Lambda = cste \tag{3.3}$$

Where

$$A_3 = 32al^2 p\rho \ln\left(\frac{s + 2l}{s - 2l}\right) - 4al\rho s\left(3d^2 + 4l^2\right)$$

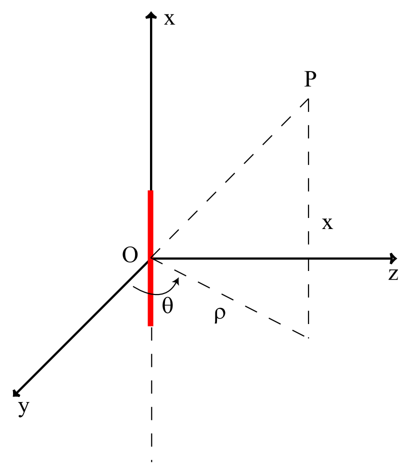

Fig. 2. *Sidereal referential and the cylindrical coordinates.*

$$A_4 = 8l^2 a \left(s^2 - 2p\right) - 3as^2 d^2 - 16l^4 a + 32bl^2$$

$$A_5 = 2l \left(xs - ld\right) \left(3d^2 - 4l^2\right) + 12lsd \left(ls - xd\right)$$

$$A_6 = s \left(xs - ld\right) \left(8l^2 - 3d^2\right) - 8l^2 x \left(s^2 - 2p\right) + 8l^3 sd - 3s^2 d \left(ls - xd\right)$$

and

$$A_7 = 8l^2 \left(s^2 - 2p\right) - 3s^2 d^2 - 16l^4 + \frac{32bl^2}{a}$$

The case of homogeneous profile of density, $a = 0$ and $b = \lambda = \frac{M}{2l}$, lead to the equations

$$\ddot{\rho} = \frac{\Lambda^2}{\rho^3} - \frac{2\mu s\rho}{p\left(s^2 - 4l^2\right)} \qquad \ddot{x} = -\frac{2\mu x}{sp}$$

We obtain the equation (3) as in Riaguas et al. (1999). In our case of inhomogeneous straight segment Eqs (3.1), (3.2) and (3.3) are strongly non linear and coupled. It need a deep numerical treatment. In fact, it is out of view to plane to work it out in an analytical way.

4 Numerical integration

To have a deep insight about the dynamical behavior of the test particle in the field of the inhomogeneous straight segment, we have to solve Eqs. (3.1), (3.2) and (3.3). In this system of differential equations the unknowns are ρ, θ and x. We derive some curves both, in the plan and in the space.
Fig. 3, Fig. 4 and Fig. 5 give some orbits in the plan and in the space corresponding to different initial conditions. We notice, in a qualitative point of view, the existence of many behavior , we obtain the state: collision, confined and not confined. More analysis about the curves below are developed in Najid et al. (2011).

5 Conclusion

In this work, we established the analytical expression of the potential generated by a straight segment with a quadratic profile of its density. This potential model in an accurate manner celestial elongated bodies in the solar system. We derived some curves (trajectories) both in two and three dimensions. They gave an overview of the dynamical behavior of massless test particle. A deep study is fulfilled in Najid et al. (2011) by using the Poincaré surface of section. After the achievement of the dynamical behavior of a test particle in the field of that segment, fixed in space, we plane, in a next future, to study the case where the segment is in rotation.

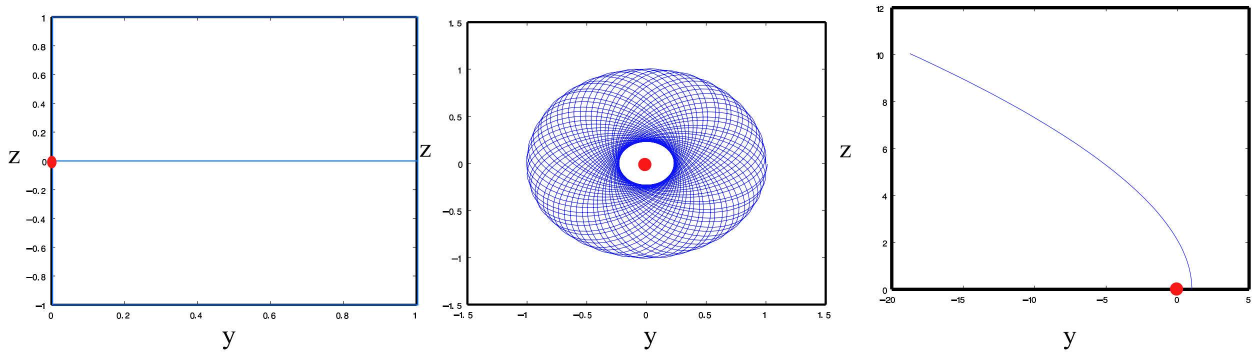

Fig. 3. *Trajectories in the plan yz .*

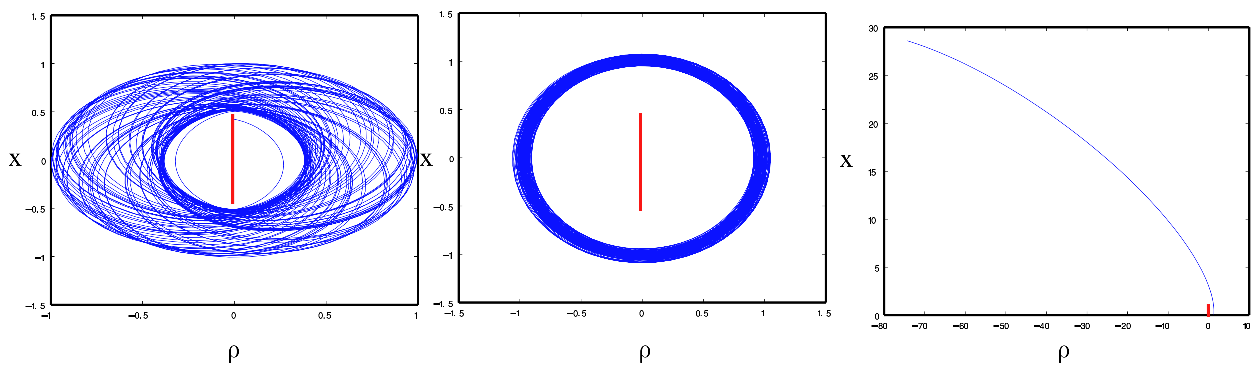

Fig. 4. *Trajectories in the plan $x\rho$.*

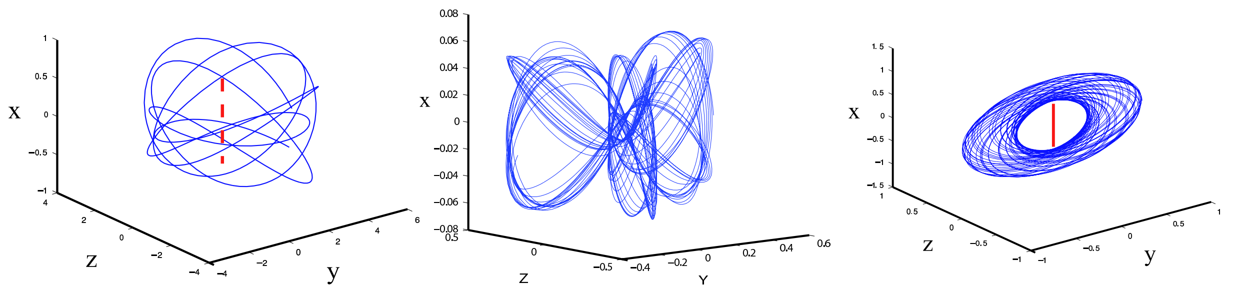

Fig. 5. *Trajectories in the space.*

References

Riaguas,A., Elipe, A., & Lara, M. 1999, Celest.Mech.Dyn. Astr, 169, 178

Elipe, A., & Lara, M. 2003, The journal of Astronautical Sciences V.51, 391, 404

Werner, R.A., & Scheeres, D.J. 1997, Celest.Mech.Dyn.Astr, 313, 343

Werner R.A. 1994, Celest.Mech.Dyn.Astr, 253, 278

Przemyslaw, B., & Slawomir,B., 2003, Celest.Mech.Dyn, 131, 141

Przemyslaw, B., Slawomir, B. & Przemyslaw, J. 2006, Celest.Mech.Dyn, 31, 48

Najid, N.-E., Elourabi, E., & Zegoumou, M. 2011, Research in Astronomy and Astrophysics, 345, 352