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DEVELOPMENT OF TECHNIQUES TO STUDY THE DYNAMIC OF HIGHLY ELLIPTICAL ORBITS

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Abstract. Many spacecrafts are or will be placed in highly eccentric orbits around telluric planets of the Solar system. Such eccentricities allow to cover a wide range of altitudes, mainly for planetology purposes. There are also orbits with very high eccentricity around the Earth, especially the GTO (Geostationary Transfer Orbit) and orbits of some space debris. In this case, the motion is strongly perturbed by the luni-solar attraction. For various reasons which will be recalled, the traditional tools of celestial mechanics are not well adapted to the particular dynamic of highly eccentric orbits. Therefore, it is necessary to develop specific techniques for this configuration. This concerns numerical as well as analytical tools. We will show how to construct the expression of the disturbing function due to the presence of an external body, well-suited for highly eccentric orbits. Expansion of the elliptic motion in closed-form by using Fourier series in multiple of the eccentric anomaly will be presented. On the other hand, classical methods of numerical integration have often a poor efficiency. We will show the interest of geometric integrators and in particular the variational integrators.

Keywords: Third-body, disturbing function, Hansen-like coefficients, elliptic motion, high eccentricity, closed-form, variational integrators

1 Introduction

When dealing with highly elliptical orbits, we have to face several difficulties. Due to the fact that such orbits cover a wide range of altitudes, the hierarchy of the perturbations acting on the satellite changes with the position on the orbit. At low altitude, the oblateness of the Earth (the so called J_2 effect) is the dominant perturbation while at high altitude the luni-solar perturbation acceleration can reach or exceed the order the J_2 acceleration. This particular configuration requires to develop adapted strategies to propagate the orbit by means of analytical theories form the one hand and numerical integration on the other hand.

From the analytical point of view, the traditional theories of celestial mechanics are not well adapted to this particular dynamic. On the one hand, analytical solutions are quite generally expanded into power series of the eccentricity and so limited to quasi-circular orbits. On the other hand, the time-dependency due to the motion of the third body is almost always neglected.

Regarding the numerical methods, the traditional integrators can be numerically unstable for high eccentricity if a moderate step size is chosen due to the very fast variation of the perturbation around the perigee. If the step size is taken extremely small this implies large round-off errors and hight CPU cost. Experiments show that even numerical integrators with variable step size does nit solve perfectly this problem.

The paper is organized as follows. In Section 2, we propose a new expression of the disturbing function of the third-body problem which is in closed form with respect to the satellite eccentricity and still permits to construct an analytical theory of the motion. We will show that the use of the eccentric anomaly instead of the mean anomaly as fast angular variable fulfills this requirement. In Section 3, we give an overview of the variational integrators and we will see the interest of using such methods rather than classical integrators for orbital mechanics problems.

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2 Third-body problem

2.1 Expression of the disturbing function of the third-body problem

Let us consider a satellite of position vector $\mathbf{r} = r\mathbf{u}$ orbiting a central body and a third body of position vector $\mathbf{r}' = r'\mathbf{u}'$, with $\mathbf{u}$ and $\mathbf{u}'$ unit vectors. The disturbing function R of the third-body problem can be expressed into spherical coordinates (r, ϕ, λ) using the traditional expansion in Legendre polynomials $P_n(x)$ as follow

$$R = \frac{\mu'}{r'} \sum_{n\geq 2} \left(\frac{r}{r'}\right)^n P_n(\mathbf{u}\cdot\mathbf{u}') \ , \tag{2.1}$$

where $\mu' = \mathcal{G}m'$, m' being the mass of the third body and $\mathcal{G}$ the gravitational constant.

In order to construct an analytical theory, it is more suitable to express (2.1) as function of orbital elements (semi-major axis a, eccentricity e, inclination I, argument of perigee ω, longitude of the ascending node Ω and mean anomaly M) or equivalent variables. From several works as Kaula (1962), Giacaglia & Burša (1980), Lane (1989) or yet Brumberg (1995), we were able to obtain a general expression of the disturbing function into Hill-Whittaker canonical variables: r, $\dot{r}$, $\theta = \omega + \nu$, $G = \sqrt{\mu a(1-e^2)}$, Ω and $H = G\cos I$, with ν the true anomaly:

$$R = \frac{\mu'}{a'} \sum_{n\geq 2} \sum_{m=-n}^{n} \sum_{m'=-n}^{n} \sum_{p=0}^{n} \sum_{p'=0}^{n} \left(\frac{a}{a'}\right)^n \left(\frac{r}{a}\right)^n \left(\frac{a'}{r'}\right)^{n+1} \mathcal{D}_{n,m,m',p,p'} \exp i\left(\Psi_{n,m,p} - \Psi'_{n,m',p'}\right) \ , \tag{2.2a}$$

$$\mathcal{D}_{n,m,m',p,p'}(I, I', \varepsilon) = (-1)^{m-m'} \frac{(n-m')!}{(n+m)!} \widetilde{F}_{n,m,p}(I) \widetilde{F}_{n,m,'p'}(I') U_{n,m,m'}(\varepsilon) \ , \tag{2.2b}$$

$$\Psi_{n,m,p} = (n-2p)\theta + m\Omega \ , \tag{2.2c}$$

$$\Psi'_{n,m',p'} = (n-2p')\theta' + m'\Omega' \ , \tag{2.2d}$$

where the $\widetilde{F}$-functions are related to the Kaula's inclination functions (see Kaula (1961)), ε is the obliquity and the U-functions are to the Wigner formula (see Sneeuw (1992)) giving the components of the rotation matrix between equatorial to ecliptic plane related.

In order to have a perturbation fully expressed in orbital elements, we expand the functions of the elliptical motion $(r/a)^n \exp i\nu$ and $(a'/r')^{n+1} \exp i\nu'$ into Fourier series with respect to an angular variable and coefficients which depend of the eccentricity. Quite generally, these functions are expanded in multiples of the mean anomaly as follow (see for example Kaula (1962), Giacaglia (1974))

$$\left(\frac{r}{a}\right)^n \exp im\nu = \sum_{s=-\infty}^{\infty} X_s^{n,m}(e) \exp isM \ , \tag{2.3}$$

where the $X_s^{n,m}$ are the so-called Hansen coefficients. In the general case, the series (2.3) always converge as Fourier series but can converge rather slowly (see e.g Brumberg & Brumberg 1999). Only in the particular case where e is small, the convergence is fast thanks to the d'Alambert property which ensure that $e^{|k-q|}$ can be factorized in $X_q^{n,k}(e)$. That is why the method is reasonably efficient for most of the natural bodies (in particular the Sun and the Moon) but can fail for satellites moving on orbits with high eccentricities. In this case, Fourier series of the eccentric anomaly E, are much more efficient :

$$\left(\frac{r}{a}\right)^n \exp im\nu = \sum_{s=-\infty}^{\infty} Z_s^{n,m} \exp isE \ , \tag{2.4}$$

where the Z-functions are called the Hansen-like coefficients. Expressions of these coefficients are given in Brumberg & Fukushima (1994) and can be computed using recurrence relations (see Lion & Métris (2011)). Using this development, we have the double advantage when $0 \leq |m| \leq n$ (which occurs in the third-body problem) that these coefficients admit a closed-form representation and that the sum (2.4) is exactly limited to $s = \pm n$ (coefficients are null for $|s| > n$).

Using the Fourier series (2.3) and (2.4) we show in Lion et al. (2011) that the disturbing function R takes the form:

$$R = \sum_{n\geq 2} \sum_{m=-n}^{n} \sum_{m'=-n}^{n} \sum_{p=0}^{n} \sum_{p'=0}^{n} \sum_{q=-n-1}^{n+1} \sum_{q'=-\infty}^{+\infty} \frac{a}{r} \mathcal{A}_{n,m,m',p,p',q,q'} \exp i\Theta_{n,m,m',p,p',q,q'} \ , \tag{2.5a}$$

$$\mathcal{A}_{n,m,m',p,p',q,q'} = \frac{\mu'}{a'}\left(\frac{a}{a'}\right)^n \mathcal{D}_{n,m,m',p,p'}(I,I',\varepsilon) Z_q^{n+1,n-2p}(e)\, X_{q'}^{-n-1,n-2p'}(e') \ , \tag{2.5b}$$

$$\Theta_{n,m,m',p,p',q,q'} = \widetilde{\Psi}_{n,m,p,q} - \widetilde{\Psi}'_{n,m',p',q'} \ , \tag{2.5c}$$

$$\widetilde{\Psi}_{n,m,p,q} = qE + (n-2p)\omega + m\Omega \ , \tag{2.5d}$$

$$\widetilde{\Psi}'_{n,m',p',q'} = q'M' + (n-2p')\omega' + m'\Omega' \ . \tag{2.5e}$$

2.2 Lie transformations perturbation method

The idea is to use a perturbative method based on the time-dependent Lie transform Deprit (1969) in order to obtain an approximated analytical solution of the third-body problem. Because the canonical variable $h = \omega$ is not ignorable and g is not automatically removed in the same time that the fast angle $l = M$ after a canonical transformation (contrary to the J_2 problem case), our initial Hamiltonian $\mathcal{H}_0$ of order 0 contains not only the keplerian energy, but also the secular part of the J_2 problem. The disturbing function R belongs to the hamiltonian $\mathcal{H}_1$ of order 1. In that way, $\mathcal{H}_0$ depends of the three momenta L, G and H which will allow to eliminate the three conjugate angles l, g and h from the transformed hamiltonian. Next, we use the homological equation providing the Lie generator W and the new Hamiltonian $\mathcal{K}$ at first order. The new Hamiltonian $\mathcal{K}$ is taken such as it does not depend on any angular variable. The Lie generator W_1 is obtained by solving a PDE involving variables which are linear with time and the eccentric anomaly which is not linear with time. Solution of W_1 can be computed with by means of two different methods. If we seek a separable solution of the PDE we find the exact solution involving Anger and Weber functions. The other method is to solve the PDE by means of a recursive process which may be more suitable for our analytical theory.

3 Variational integrators

3.1 Philosophy

Variational integrators derive from a discrete version of the least action principle. Instead of a continuous path $q(t)$ for $t \in [t_i, t_f]$, we consider a discrete path $q : \{t_0 = t_i, t_1, \ldots, t_k, \ldots, t_N = t_f\}$ where $k, N \in \mathbb{N}$, q_k being an approximation to $q(t_k)$. Hence, the Lagrangian $L(q, \dot{q}, t)$ of the system is approximated on each time interval $[t_h, t_{k+1}]$ by a discrete Lagrangian $L_d(q_k, q_{k+1}, h)$, with $h = t_{k+1} - t_k$ being the time interval. For conservatives systems, we just compute the principle of discrete stationary action, which gives the discrete Euler-Lagrange (DEL) equations and the discrete Hamilton equations (see West (2004)):

$$p_k = -D_1 L_d(q_k, q_{k+1}) \quad , \quad p_{k+1} = D_2 L_d(q_k, q_{k+1}), \tag{3.1}$$

where $D_i L$ denotes the derivative of L with respect to the i slot. For (q_k, p_k) known, we can compute q_{k+1} then p_{k+1}.

In case of dissipative or forced systems, the discrete action can be modified by adding the non-conservative force and using the Lagrange d'Alembert principle.

The variational integrators preserve the geometric structure of the mechanical system. This has two consequences. Firstly, the schemes are symplectic and so, we have a good energy behaviour for equal time steps. Secondly, momenta and symmetries are conserved (via the discrete Noether's theorem). More over, one can obtain higher-order methods by using higher-order quadrature to approximate the Lagrangian (e.g, Gauss-Lobatto) as our variational integrator RKN6 tested in the following section.

3.2 Numerical tests for conservative system

To illustrate the performance of the variational integrators we consider the keplerian problem which is a simple and an integrable system. The Lagrangian describing this problem is

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T \dot{q} - V(q) \quad \text{and} \quad V(q) = -\frac{\mu}{\|q\|} \ , \tag{3.2}$$

where $q = q(t) \in \mathbb{R}^2$ is the trajectory of body orbiting the primary body (Earth) and μ is the gravitational parameter. The equations of motion are given by the Euler-Lagrange equations and the trajectories of this conservative system have two conserved quantities: the energy of the system and the total angular momentum.

So, the easiest way to check on the reliability of integration schemes is to watch the behavior of these conserved quantities. The algorithms for which we present results are:

- RK6: Explicit Runge-Kutta 6-order, with fixed stepsize h=20s and h=60s.
- Symmetric RKN6*: Symmetric and reversible Runge-Kutta Nyström 6-order, using the 4-stage Lobatto IIIA† method, with fixed stepsize h=60s (see Cash & Girdlestone (2006)).
- Variational RKN6: Variational Runge-Kutta Nyström 6-order, using the Gauss-Lobatto quadrature rules with a 4-stage Lobatto IIIA method and a fixed stepsize h=60s. This integrator was built from the papers of Farr & Bertschinger (2007) and Farr (2009).
- Ode113: variable order Adams-Bashforth-Moulton (PECE) solver in Matlab with adaptative stepsize.

As initial conditions, we have chosen an highly elliptical orbit with: a = 36890.683 km, e = 0.8. The integration is performed over 325 days corresponding to 400 orbits.

The relative error in the energy for each integrators is plotted in Fig. 1. As expected, we can see that the classical integrators RK6 and ode113 do not preserve the energy of the system, even if ode113 uses an adaptative stepsize. The variational RKN6 preserves better energy than the symmetric and reversible RKN6.

Results for the relative error in the total angular momentum behaviour lead to the same conclusion. The standard integrators have a divergent angular momentum, while the others it is conserved with at most the finite fluctuations.

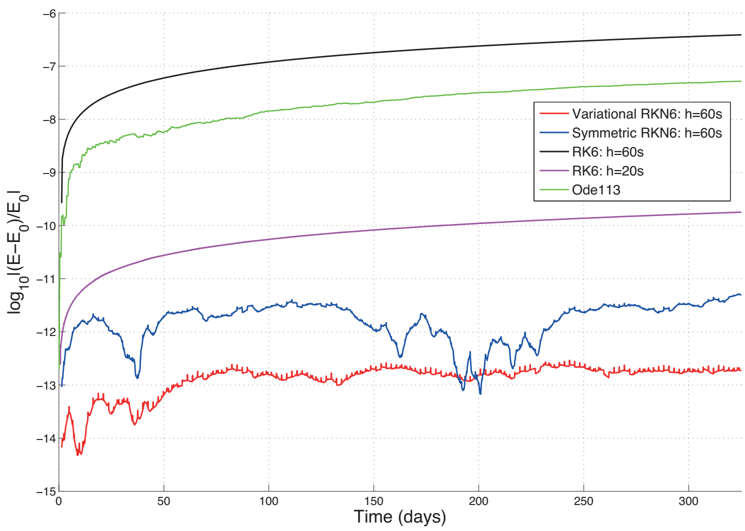

Fig. 1. Relative error in the energy behaviour of integrators for a conservative system

Conservation properties are not the only indicators for the quality of integrators. The amount of computational resources consumed in this process is equally important, as any algorithm can be trimmed to produce highly accurate results. Using for the variational integrators a predictor of high order simply derived from finite difference methods, we have reduced up to 35 percent the number of function evaluation.

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*Source code available on `http://www2.imperial.ac.uk/~jcash/GI_software/readme.php`.

†The Lobatto IIIA are collocation methods symmetric and symplectic (see Hairer et al. (2002)).

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