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THE GRAVITATIONAL POTENTIAL OF AXIALLY SYMMETRIC BODIES FROM A REGULARIZED GREEN KERNEL

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Abstract. The determination of the gravitational potential inside celestial bodies (rotating stars, discs, planets, asteroids) is a common challenge in numerical Astrophysics. Under axial symmetry, the potential is classically found from a two-dimensional integral over the body's meridional cross-section. Because it involves an improper integral, high accuracy is generally difficult to reach. We have discovered that, for homogeneous bodies, the singular Green kernel can be converted into a regular kernel by direct analytical integration. This new kernel, easily managed with standard techniques, opens interesting horizons, not only for numerical calculus but also to generate approximations, in particular for geometrically thin discs and rings.

Keywords: Gravitation, Methods: analytical, Celestial mechanics

1 Introduction

According to Newton's law, the potential ψ associated with two point masses at a relative distance δr from each other diverges as $\delta r \to 0$. As a result, for any continuous body, the gravitational potential is obtained by integrating a function which diverges everywhere in the inside. The expansion of the Green kernel into Legendre polynomials (which circumvent this singularity problem), as usually done, leads to numerical oscillations due to truncations (Clement 1974). When the body is axially symmetric, the integration over the polar angle ϕ results in the well known expression (e.g. Durand 1953):

$$\psi(R,Z) = -G\int_{\text{volume}} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} da\, dz\, a\, d\phi \;\longrightarrow\; -2G\int_{\text{section}} \rho(a,z)\sqrt{\frac{a}{R}}\, k\, \mathbf{K}(k)\, da\, dz \tag{1.1}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind, k is the modulus, with $(a+R)^2+(z-Z)^2 = 4aR/k^2$. The singularity is still present here since $|\vec{r}-\vec{r}'| \to 0$ now corresponds to $k \to 1$. Figure 1 shows the logarithmic divergence of the function $\mathbf{K}(k)$ near unity. Unfortunately, there is no way to obtain a more advanced form for the above bidimensional integral, except for a few special cases. For instance, in the absence of any radial density gradient, the integration along the radial direction can be performed analytically. This fundamental result, described in Durand (1953) and rediscovered in the astrophysical context by Lass & Blitzer (1983), is:

$$\psi(R,Z) = -G\int_{\text{boundary}} \rho(z)\left[4\sqrt{aR}\frac{\mathbf{E}(k)}{k} + \frac{a^2-R^2-(z-Z)^2}{\sqrt{aR}} k\mathbf{K}(k) - (z-Z)\Omega\right] dz \tag{1.2}$$

where $\mathbf{E}$ is the complete elliptic integral of the second kind, and Ω is the solid angle sustained by the disc (radius a, altitude z) when seen from point (R,Z). Actually, this expression is nothing but the total potential due to a collection of infinitely thin circular plates (i.e. discs) piled up along the z-direction (see Fig. 2a). This kind of formula is therefore well suited not only for homogeneous bodies but also for inhomogeneous bodies such that $\partial_a \rho = 0$. The integrand in Eq.(1.2) being regular, there is no difficulty to get accurate potential values.

There is apparently no equivalent form of Eq.(1.2) corresponding to bodies having zero vertical density gradients. In other words, the potential due to a semi-infinite homogeneous cylinder might not exist in a closed form. This question is the subject of a longstanding debate. As a matter of fact, the potential and forces due

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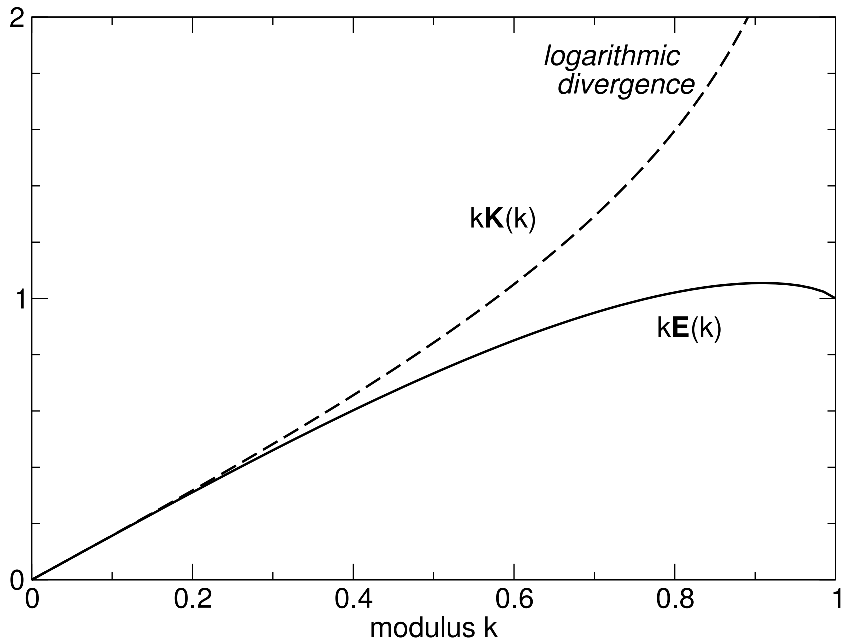

Fig. 1. Part of the initial Green kernel $k\mathbf{K}(k)$ and part of the new regular kernel $k\mathbf{E}(k)$; see Eqs.(1.1) and (2.1).

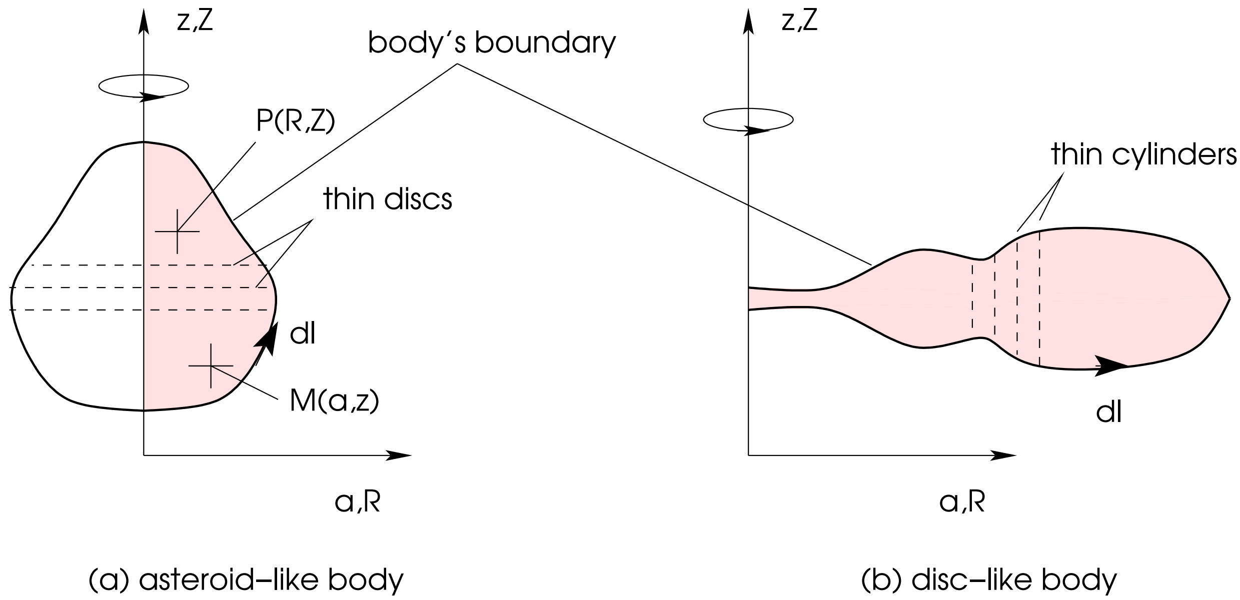

Fig. 2. Two different configurations of axially symmetric bodies, with their geometrical cross-section where $\rho \neq 0$ (shaded zone) and boundary (bold).

to charged, toroidal structures are required in many contexts of research including Biology, Electromagnetism and Plasma physics. In his remarkable textbook, Durand (1953) mentions that "*[...]Le calcul analytique de l'intégrale du potentiel paraît difficile [...]*". He can only give a solution in the form of infinite series involving Legendre polynomials. This series is however an alternate series which poorly converges, as usually observed (Clement 1974). A constant and careful bibliographic search shows that this question is still open.

2 A new regularized kernel ?

After some effort and investment in integral calculus involving elliptic integrals, we have recently discovered a simple way to bypass the singularity of Green kernel precisely in the case where $\partial_z \rho = 0$. The result is based on the derivation of this following new definite integral:

$$\int k\mathbf{K}(k)dz = \int k\mathbf{E}(k)dz + (z-Z)k\left[\mathbf{K}(k) - m'^2\mathbf{\Pi}(m,k)\right], \tag{2.1}$$

where $\mathbf{\Pi}(m,k)$ is the complete elliptic integral of the third kind, $m = 2\sqrt{aR}/(a+R)$ is the characteristic, and m' is its complementary (i.e. $m'^2 + m^2 = 1$). All the details of the derivation will be presented in a forthcoming paper. The first term in the right-hand-side of Eq.(2.1) is fully regular since $\mathbf{E}(k) \in [\frac{\pi}{2}, 1]$. Figure 1 compares the two kernels $k\mathbf{K}(k)$ and $k\mathbf{E}(k)$. The second term is also fully regular: the divergence of the $\mathbf{K}$- and $\mathbf{\Pi}$-functions as k and m approach unity is cancelled out by the presence of the two vanishing factors m' and $(z-Z)$. The potential of any vertically homogeneous axially symmetric body is then found from the expression:

$$\psi(R,Z) = -2G\int_{\text{boundary}} \rho(a)\sqrt{\frac{a}{R}}\left\{(z-Z)k\left[\mathbf{K}(k) - m'^2\mathbf{\Pi}(m,k)\right]\right\}da - 2G\int_{\text{section}} \rho(a)\sqrt{\frac{a}{R}}k\mathbf{E}(k)dzda \tag{2.2}$$

This expression is exact. It is interpreted as the total potential due to a collection of infinitely thin (semi-infinite) coaxial cylinders* side by side along the a-direction (see Fig. 2b). In some sense, Eqs.(1.2) and (2.2) are complementary: the use of the one or the other depends on the shape of the body and on it density structure as well.

3 Note on Green's theorem for fully homogeneous bodies

For fully homogeneous bodies, we have $\partial_a \rho = \partial_z \rho = 0$, and so Eqs.(1.2) and (2.2) are formally equivalent. The ability to integrate analytically Eqs.(1.2) or (2.2) must be considered (this is for instance possible for the homogeneous sphere). In this purpose, the conversion of the double integral in the right-hand-side of Eq.(1.1) into a line (or contour) integral is possible by using Green's theorem, provided the kernel is the curl of a certain field $\vec{F}$. Actually, for any vector field $\vec{F}(F_a, F_z)$, Green's theorem writes in the present context:

$$\int_{\text{section}} \nabla \times \vec{F} dadz \cdot \vec{u}_\phi = \oint_{\text{boundary}} \vec{F} \cdot \vec{d\ell} \tag{3.1}$$

where $\vec{d\ell}$ is an infinitesimal displacement along the boundary (oriented counter-clockwise). This powerful approach is stressed in Ansorg et al. (2003) who compute new figures of equilibrium and bifurcations from the Maclaurin sequence. Indeed, they have determined a vector $\vec{F}$ such that:

$$\sqrt{\frac{a}{R}}k\mathbf{K}(k) \propto \nabla \times \vec{F} \cdot \vec{u}_\phi = \frac{\partial F_z}{\partial a} - \frac{\partial F_a}{\partial z} \tag{3.2}$$

and so, from Eq.(1.1), they immediately get the potential, namely:

$$\psi(R,Z) \propto \int_{\text{boundary}} (F_a da - F_z dz). \tag{3.3}$$

This allows to derive the potential everywhere in space through a single integral, which is numerically very advantageous. In practice, it seems however difficult to reach high accuracy when the potential is required just onto the boundary (or contour), and so the numerical treatment must still be faced with care (Ansorg et al. 2003).

*For finite size bodies, one must subtract the contribution of two semi-infinite cylinders, after shifting the one with respect to the other.

4 Concluding remarks

The new, regular kernel appearing in Eq.(2.2) is a substitute for the genuine Green kernel whose singularity avoids, from the numerical point of view, any direct treatment. This kernel is particularly interesting in the context of geometrically thin discs and rings. For such objects actually, the radial and vertical structures are often decoupled (Shakura & Sunyaev 1973) and the density of gas is uniform in the direction perpendicular to the equatorial plane (but variable in radius). The formula is indeed helpful at the numerical level even for axially symmetric bodies having radial and vertical density gradients in the framework of the splitting method (Huré 2005). It is also useful at the theoretical level, for instance to derive reliable approximations for the mid-plane gravitational potential and acceleration. In this case for instance, we have $k \approx m$ at the first order in $z/(a+R)$. Thus, we have simply:

$$\int_{\text{section}} \rho(a)\sqrt{\frac{a}{R}}k\mathbf{E}(k)dzda \approx \int_{\text{boundary}} \Sigma(a)\sqrt{\frac{a}{R}}m\mathbf{E}(m)da \tag{4.1}$$

where Σ is the local surface density. More generally, it would be interesting to convert the section integral in Eq.(2.2) into a contour integral via Green's theorem. This question is currently under study.

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