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APPROXIMATE SOLUTION FOR THE GRAVITATIONAL POTENTIAL OF THIN DISKS

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Abstract. We are interested in the derivation of reliable formulae for the gravitational potential of disks, under the assumption of axial symmetry. As a consequence of the Newton's law, the formula contains a diverging kernel which is always difficult to manage. In the particular case of vertically homogeneous bodies, we have built an equivalent kernel which is free of singularity, and therefore, well suited to numerical computations. From this new expression, in the case of geometrically thin disks, we have derived a good approximation. This formula reproduces the potential inside the body with a relative error as low as 10^{-3} typically. We will see, through various torus shapes, that our approximation can be used for thin and relatively thick disks.

Keywords: disks, Gravitation, Methods: numerical

1 Introduction

Thin disks are found in various types of objects in the Universe: around planets, in forming stars and in galaxies. As they host a certain amount of matter, they naturally generate a gravitational potential which can influence their own structure and dynamics, especially if they are "massive enough", compared to their environment (central star, circumstellar envelope, bulge, etc.). It is therefore important to build reliable formulae enabling to analyze various aspects of disk self-gravity, and the natural starting point to this is Newton's law. This law implies that the gravitational potential of matter enclosed in a volume V writes:

$$\Psi(\vec{r}) = \int_V \frac{\rho(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} d^3\vec{r'}, \tag{1.1}$$

and this is basically the convolution product of the mass density $\rho(\vec{r'})$ with the Green function $1/\mid \vec{r} - \vec{r'} \mid$. In practice, the potential is difficult to compute due to the hyperbolic divergence of the Green function when $\vec{r} \to \vec{r'}$, and this often supports the use of multipole expansions. For an axially symmetric object, the potential takes the form (Durand 1953):

$$\begin{cases} \psi(R,Z) = -2G \int_S \rho(a,z) \sqrt{\frac{a}{R}} k\mathbf{K}(k) da dz, \\ k = \frac{2\sqrt{aR}}{\sqrt{(a+R)^2+(Z-z)^2}}, \end{cases} \tag{1.2}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind, k is the modulus, and other variables are cylindrical coordinates (a typical configuration is shown in Fig. 1). This is a double integral over the cross-section S and the function $2\sqrt{\frac{a}{R}}k\mathbf{K}(k)$ plays the role of an "axially symmetrical Green function". It is particularly well suited to toroidal systems like disks and rings. As a direct benefit of the integration over the polar angle, the divergence of this new Green function is weaker: it occurs for $k \to 1$ (i.e for $\vec{r} \to \vec{r'}$), but it is now logarithmical. This does not render the numerical computation much easier. A common method is to expand this axially symmetrical Green function into Legendre polynomials (Hachisu 1986). This effectively avoids the divergence problem but generates new difficulties such as series truncations and convergence rate (Clement 1974).

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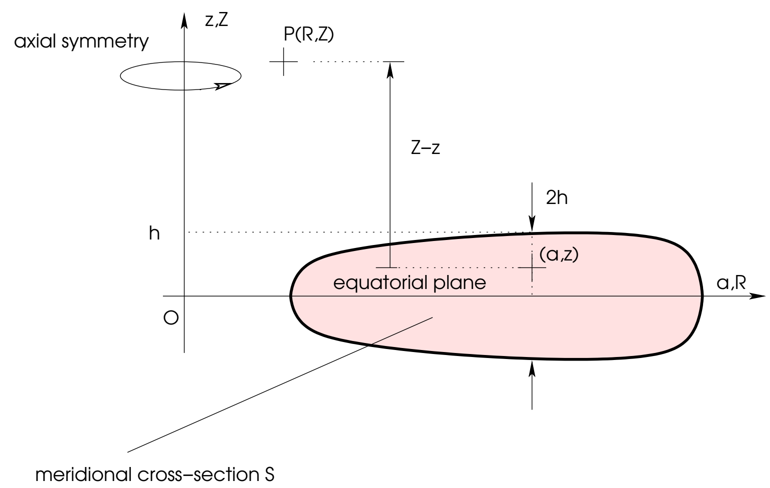

Fig. 1. Typical configuration for the gravitating, axially symmetric disk (finite size and mass, and local total thickness $2h$), and notations associated with the cylindrical coordinate system.

2 A general expression for the newtonian potential of vertically homogeneous bodies

Assuming that the mass density is, at a given radius a, independent on the altitude (i.e. there is no density stratification perpendicularly to the equatorial plane), then it is easy to rewrite the above expression as

$$\begin{cases} \psi(R,Z) = -G\int_a \mathcal{G}\Sigma da, \\ \\ \mathcal{G} \equiv \frac{1}{2h}\int_z 2\sqrt{\frac{a}{R}} k\mathbf{K}(k)dz \equiv \mathcal{G}(a,h;R,Z), \end{cases} \tag{2.1}$$

where

$$\Sigma = \int_z \rho(a)dz = 2\rho(a)h(a), \tag{2.2}$$

is the local surface density, h is the local semi-thickness (possibly a function of the radius), and $\mathcal{G}$ is the axially symmetrical Green function averaged over the disk thickness at the actual radius. This "mean Green function" is a purely geometrical function which depends on a, on the local thickness and the coordinates (R, Z) where the potential is requested. As shown in Trova et al. (2012), we can rewrite the mean Green function in the following form (see also Trova et al. 2011):

$$\begin{cases} \mathcal{G}\left(R,Z;a,h\right) = \frac{1}{h}\sqrt{\frac{a}{R}}\left[\int_{-h}^{h} k\mathbf{E}(k)dz - (Z-h)\mathbf{H}(m^2,k_+) + (Z+h)\mathbf{H}(m^2,k_-)\right], \\ \\ \mathbf{H}(m,k) = k\left[\mathbf{K}(k) - m'^2\mathbf{\Pi}(m^2,k)\right], \end{cases} \tag{2.3}$$

where $\mathbf{E}$ and $\mathbf{\Pi}$ are the complete elliptic integrals of the second and third kinds respectively, $k_\pm = k(z = \pm h)$, $m \equiv k(z = Z)$ and $m'^2 = 1 - m^2$. This new form is very interesting first because each term of Eq.(2.3) is a regular function even when the modulus is unity. Actually, the divergence of the $\mathbf{H}$ function when $k_\pm = 1$ is cancelled by the presence of the multiplying factor $Z \mp h$ which is then zero. This means that the radial integral in Eq.(2.1a) using Eq.(2.3) can be performed by standard numerical techniques. Second, we have made no kind of approximation in deriving Eq.(2.3) meaning that the properties of the Newton's law are automatically saved everywhere in space. Third, the formula works for any disk shape.

3 Approximation for the potential of geometrically thin disks and rings

We see from Eqs.(2.1a) and (2.3) that the determination of the potential still requires a double integral over S. In the case of geometrically thin disks and rings (those systems where the thickness is locally small compared to the radius; see e.g. Shakura & Sunyaev 1973; Pringle 1981), we can however get a good approximation for $\mathcal{G}$, leaving finally a single integral over a. This is achieved by considering an expansion of the term $k\mathbf{E}(k)$ around the mean modulus $\tilde{k}$ defined by:

$$\tilde{k} = \frac{k_+ + k_-}{2} \tag{3.1}$$

and which never reaches unity. From a Taylor expansion at the first order, we actually find:

$$-\int_z k\mathbf{E}(k)dz \approx \mathbf{T}_1(\tilde{k})(Z-z) + 2\sqrt{aR}\mathbf{T}_2(\tilde{k})\,\mathrm{argsh}\left(\frac{Z-z}{a+R}\right) \tag{3.2}$$

where we have set:

$$\begin{cases} \mathbf{T}_1(k) = k\left[\mathbf{K}(k) - \mathbf{E}(k)\right], \\ \\ \mathbf{T}_2(k) = 2\mathbf{E}(k) - \mathbf{K}(k). \end{cases} \tag{3.3}$$

So, if we replace Eq.(3.2) in Eq.(2.3a), we obtain an approximation of the axially symmetrical Green mean function, namely:

$$\begin{aligned} \mathcal{G} \approx 2\sqrt{\frac{a}{R}}\Bigg\{\mathbf{T}_1(\tilde{k}) - \mathbf{T}_2(\tilde{k})\frac{\sqrt{aR}}{h}\left[\mathrm{argsh}\left(\frac{Z-h}{a+R}\right) - \mathrm{argsh}\left(\frac{Z+h}{a+R}\right)\right] \\ -\frac{Z-h}{2h}\mathbf{H}(m,k_+) + \frac{Z+h}{2h}\mathbf{H}(m,k_-)\Bigg\} + \mathcal{O}\left(\frac{h^4}{16a^4}\right) \equiv \mathcal{G}_{\mathrm{app.}} + \mathcal{O}\left(\frac{h^4}{16a^4}\right). \end{aligned} \tag{3.4}$$

and it is expected to work well for disk with small aspect ratios h/a. As a consequence, the gravitational potential of a thin three-dimensional disk is now given by a one-dimensional integral over the radius, namely:

$$\psi(R,Z) \approx -G\int_a \mathcal{G}_{\mathrm{app.}}(a,h;R,Z)\Sigma(a)da \equiv \psi_{\mathrm{app.}}(R,Z). \tag{3.5}$$

This approximation is in principle valid in the whole physical space. It can be used to model the internal structure of self-gravitating disks and rings. So, it is very attractive from a numerical point of view.

4 Checking the approximation

4.1 A torus with circular cross-section

To check the quality of the approximate potential, we have compared Eq.(2.1a) using Eq.(2.3) and Eq.(3.5) for circular, homogeneous torus. This case has been recently considered by Bannikova et al. (2011). Here, the center of the torus is at $(0.75, 0)$, and the radius is $L = 0.25$ (diameter $2L$). Figure 2a shows the relative deviation between the potential values when computed in the half-plane $(R, Z \geq 0)$. We see that the relative error is of the order of 10^{-3} inside the torus as well as outside, and reaches 10^{-4} locally. The error map is rather uniform. This indicates that we have a very good estimate of the quantity $\int k\mathbf{E}(k)dz$ for any R and Z, at least for vertically homogeneous systems. Interestingly enough, the error is especially low while the torus considered has a rather large aspect ratio h/a which reaches $1/3$ at the center. The approximation works well because the error is, as indicated above, depends on $(h/a)^4$. We conclude that our approximation remains satisfactory for geometrically thick disks, as long as $h/a \leq 1$.

4.2 Varying the torus cross-section

We have tested the limits of the approximation by varying independently the radial extension $2L$ and the vertical extension $2h$ of the torus considered before, while maintaining the center of the torus at $(0.75, 0)$. The exact potential and the approximate one are then compared at the center. The results are displayed in Fig. 2b. We find that the relative error for the potential at the center is not very sensitive to the radial extension of the torus, and the relative accuracy increases when h decreases. This means that the best objects this approximation is made for are thin disks and rings.

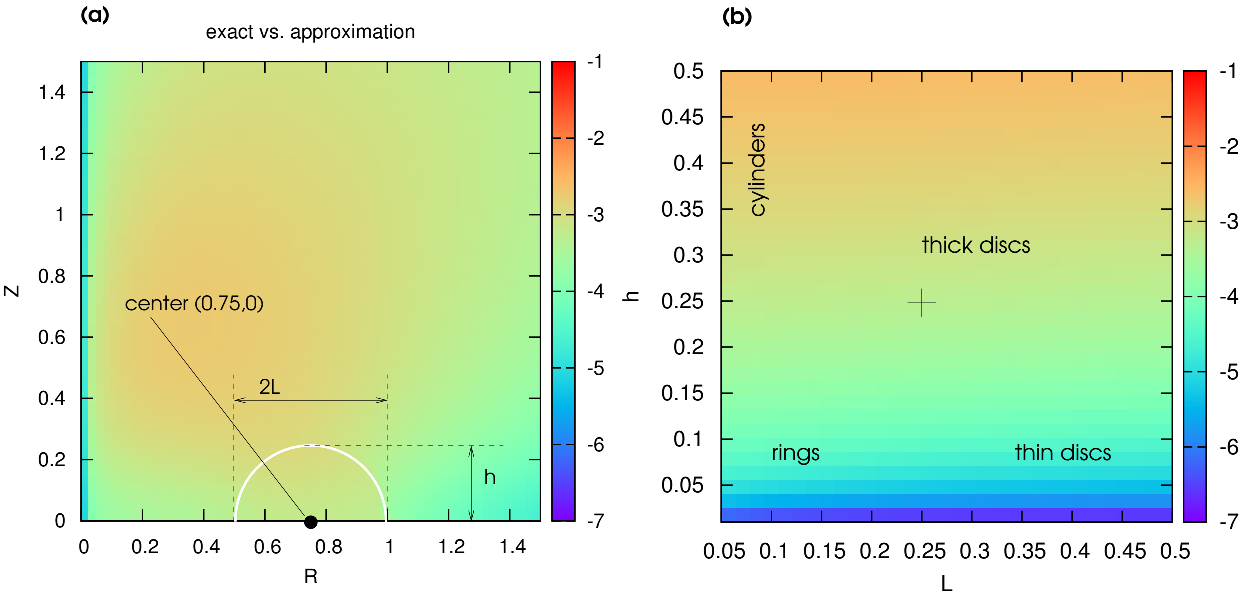

Fig. 2. Left: relative error (decimal log. scale) between the approximate potential given by Eq.(3.5) and the exact potential given by Eq.(2.1a) using Eq.(2.3) in the $(R, Z > 0)$-plane of a homogeneous torus with circular cross-section (boundary shown in white). **Right:** comparison of potential values at the center $(0.75, 0)$ of the torus when varying the diameter $2L$ and the vertical extension $2h$ of the torus (the cross corresponds to the circular torus shown left).

5 Conclusion

We have produced an equivalent expression for the gravitational potential of axially symmetrical, vertically homogeneous bodies which is free of kernel singularity. It saves the properties of Newton's law and is valid whatever the body's shape. We have also derived a reliable approximation which is especially accurate for thin disks and rings (although it works well for thick disks). This approximate formula is also numerically interesting because the potential is expressed by a radial (i.e. one dimensional) integral. Our approximation is in principle restricted to vertically homogeneous systems, and as we know, realistic systems have density gradients in three directions. We have then checked the sensitivity of the results to the $\rho(z)$ profile. As demonstrated in Trova et al. (2012), our approximate Green function gives surprisingly good results in the case of a Gaussian distribution.

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