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S. Boissier, P. de Laverny, N. Nardetto, R. Samadi, D. Valls-Gabaud and H. Wozniak (eds)

TURBULENT MIXING IN STELLAR RADIATIVE ZONES

V. Prat [1, 2] and F. Lignières [1, 2]

Abstract. Macroscopic and turbulent motions created by rotation can significantly affect internal structure and evolution of stars in a way that is currently not well understood. In particular, turbulent mixing coefficients that are currently taken into account in many stellar evolution codes have been derived on phenomenological arguments and have not been tested through numerical simulations or laboratory experiments.

Our purpose is to test the vertical turbulent diffusion coefficient generated by radial differential rotation of Zahn (1992) with 3D local direct numerical simulations of stably-stratified homogeneous stationary sheared turbulence. To reach the high thermal diffusivity regime found in stellar radiative zones, we use an asymptotic development of the Boussinesq equations in the domain of the small Péclet numbers.

We present the results of simulations performed at different turbulent Péclet numbers (including one within the small-Péclet-number approximation). These results show that the form of the vertical turbulent diffusion coefficient initially proposed by Zahn is valid for Péclet numbers smaller than one. We also give a first quantitative estimate of this coefficient through numerical simulations.

Keywords: diffusion, hydrodynamics, rotation, stellar interiors, stars

1 Introduction

Macroscopic motions induced by rotation like meridional circulation or differential rotation actively contribute to chemical mixing by forcing turbulent motions. Such a mixing has an influence on the internal structure and the evolution of a star, notably by providing fresh combustible to the core, which increases the lifetime of the star. For more information about the effects of rotation on stellar evolution, see the recent review by Maeder & Meynet (2012).

Turbulent motions involved in rotating stars are full 3D motions and thus cannot be resolved by 1D or 2D star models. To make stellar evolution models more realistic, it is therefore crucial to reliably model these motions so that they can be integrated in such evolution models. Many stellar evolution codes currently use prescriptions for turbulent mixing based on Zahn (1992) which introduces various turbulent diffusion coefficients.

Among them is the vertical turbulent diffusion coefficient D_v defined by:

$$D_\mathrm{v} = \frac{1}{3}\kappa Ri_\mathrm{cr}\left(\frac{r\sin\theta}{N_\mathrm{T}}\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 , \tag{1.1}$$

which can be put in the form:

$$D_\mathrm{v} = \frac{1}{3}\kappa\frac{Ri_\mathrm{cr}}{Ri} , \tag{1.2}$$

where κ is the thermal diffusivity of the fluid, Ri the Richardson number which compares the effects of stratification (through the Brunt-Väisälä frequency N_T) and vertical shear ($r\sin\theta\mathrm{d}\Omega/\mathrm{d}r$ in spherical geometry with a rotation profile $\Omega(r)$). Ri_cr is the critical Richardson number for which the flow is marginally stable. The purpose of this work is to test the form of this coefficient in the domain of high thermal diffusivities present in stellar radiative zones and possibly propose a model of turbulent mixing to be included in stellar evolution codes. To achieve this, we performed 3D local direct numerical simulations (DNS) of a stably-stratified sheared flow.

[1] CNRS, IRAP, 14, avenue Édouard Belin, F-31400 Toulouse, France
[2] Université de Toulouse, UPS-OMP, IRAP, Toulouse, France

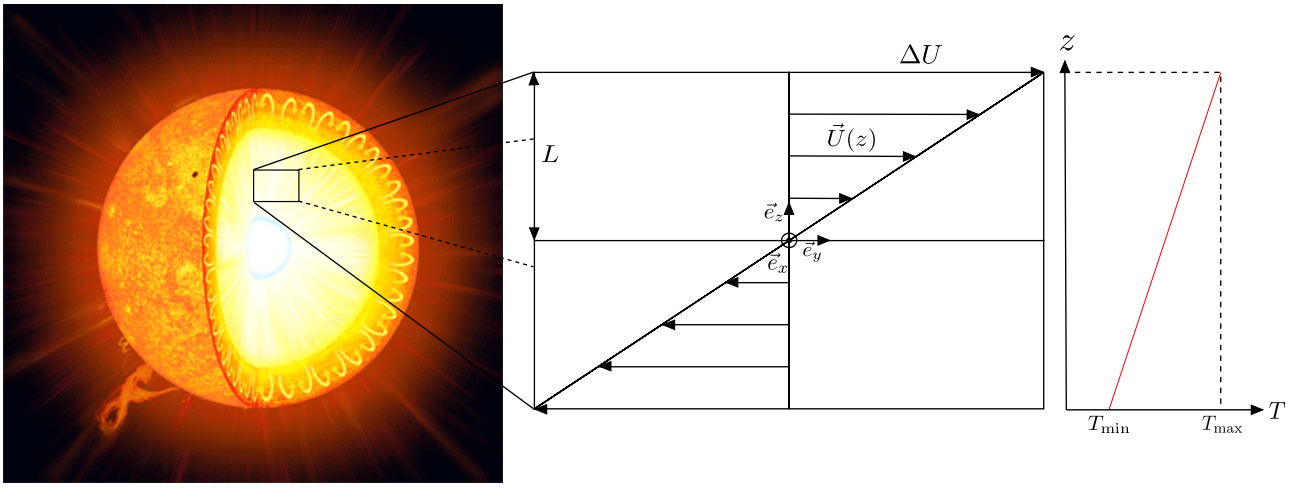

Fig. 1. Sketch of the flow

2 Theoretical framework

Figure 1 shows our cartesian numerical domain in which the local effect of differential rotation is represented by a uniform forced mean vertical shear and a stable uniform forced mean temperature gradient mimics the stable entropy stratification typical of radiative zones. Indeed, temperature and entropy are equivalent in the Boussinesq formalism. The non-dimensional governing equations read:

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{2.1}$$

$$\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}p + Ri\theta\vec{e}_z + \frac{1}{Re}\Delta\vec{v} + \vec{f}_{\rm v} \tag{2.2}$$

$$\frac{\partial\theta}{\partial t}+\vec{v}\cdot\vec{\nabla}\theta + v_z = \frac{1}{Pe}\Delta\theta + f_{\rm T} \tag{2.3}$$

where we use L, ΔU and $\Delta T = (T_{\max} - T_{\min})/2$ as respectively length, velocity and temperature scales (see Fig. 1). Velocity of the flow is noted $\vec{v}$, pressure deviation from equilibrium p and temperature deviation from the mean profile θ. Forcing terms $f_{\rm v}$ and $f_{\rm T}$ are used to impose the mean velocity and temperature profiles.

Three non-dimensional parameters appear in these equations including the already presented Richardson number and two others: the Reynolds number $Re = UL/\nu$ and the Péclet number $Pe = UL/\kappa$ where ν is the viscosity of the fluid.

The high thermal diffusivity present in stellar radiative zones creates a huge gap between the dynamical time $\tau_{\rm dyn} = L/\Delta U$ and the diffusive one $\tau_{\rm diff} = L^2/\kappa$. It is therefore impossible to perform simulations in such conditions without some sort of approximation. Nevertheless, an asymptotic version of the Boussinesq equations can be derived using a Taylor-expansion of all variables at first order in Pe (Lignières 1999). Equations (2.2) and (2.3) then become:

$$\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}p + RiPe\psi\vec{e}_z + \frac{1}{Re}\Delta\vec{v} + \vec{f}_{\rm v} \tag{2.4}$$

$$v_z = \Delta\psi \tag{2.5}$$

with $\psi = \theta/Pe$. This is called the small-Péclet-number (SPN) approximation. There are two non-dimensional parameters left in the new equations: the Reynolds number and the product of the Richardson and Péclet numbers. It means that stratification and thermal diffusion are merged into a single physical effect.

3 Simulations of turbulent transport

In order to obtain an homogeneous and stationary turbulence, as implicitely assumed in the derivation of Eq. (1.1), it is necessary to determine the value of the critical Richardson number for each considered value of the Péclet number. Then, we use two approaches to study the turbulent transport: one Lagrangian and the other Eulerian.

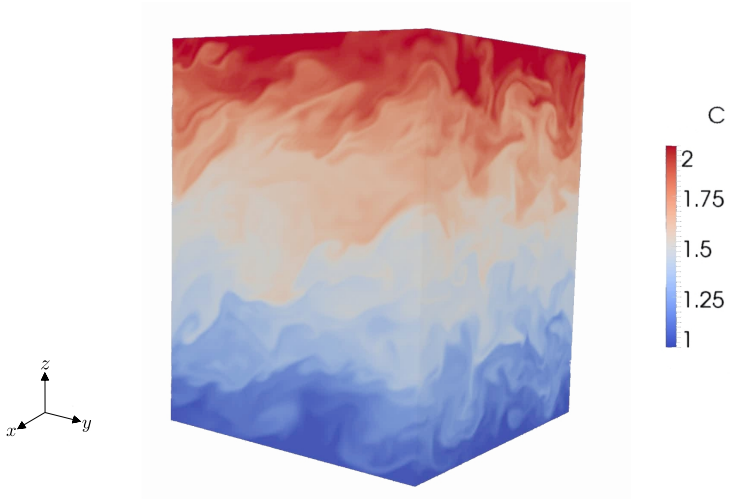

Fig. 2. Snapshot of the concentration field

3.1 Lagrangian approach

The principle of this approach is to put particles uniformly in the fluid at a given time and to follow their trajectories under the effect of the velocity field. The Eulerian homogeneity and the stationarity of our turbulence ensures its Lagrangian homogeneity, which means that the particles see the same turbulent properties during the whole simulation. In this case, Taylor (1921) shows that for times much greater than the turbulent correlation time, the transport is diffusive, the vertical mean square displacement of particles is then linear in time:

$$\left\langle [z(t) - z(0)]^2 \right\rangle = 2D_{\rm t}t \tag{3.1}$$

and the theory also gives an explicit expression for the diffusion coefficient:

$$D_{\rm t} = \int_0^{+\infty} \langle v_z(t) v_z(t+\tau) \rangle {\rm d}\tau = \langle {v_z}^2 \rangle T_{\rm L} \tag{3.2}$$

3.2 Eulerian approach

This method consists in solving an advection/diffusion equation for a passive scalar field (a concentration c) which presents a uniform mean vertical gradient Γ:

$$\frac{\partial c}{\partial t} + \vec{v} \cdot \vec{\nabla} c = D_{\rm m} \Delta c + f_{\rm c} \tag{3.3}$$

where $D_{\rm m}$ is the molecular diffusivity and $f_{\rm c}$ a forcing term. The vertical turbulent diffusion coefficient $D_{\rm t}$ is then determined *via* the relation:

$$D_{\rm t} = -\frac{\langle c v_z \rangle}{\Gamma} \tag{3.4}$$

An example of simulation with such a concentration field is shown in Fig. 2.

3.3 Comparison of the methods

The values of $D_{\rm t}$ obtained using the different methods are found to be similar, but show a significant dispersion ($\pm 20\%$). This dispersion appears to be due to temporal fluctuations of the physical properties of the flow. It is therefore natural to want to study the turbulent transport for a longer time in order to reduce these fluctuations. Whereas in the Lagrangian approach the integration time is limited by the presence of the lower and upper boundaries of the fluid, it can be arbitrarily fixed in the Eulerian approach, which makes it more appropriate.

Table 1. Results of the different runs ($S = \Delta U/L = \mathrm{d}U/\mathrm{d}z$)

Pe_ℓ	Re_ℓ	$Ri_{\rm cr}$	$(RiPe_\ell)_{\rm cr}$	$ST_{\rm L}$	$\langle {v_z}^2\rangle/\Delta U^2$	$D_{\rm t}/(SL^2)$	$\beta = D_{\rm t}/(u\ell)$
52	260	0.124	6.45	0.771	0.00350	0.00270	0.104
0.34	340	1.27	0.432	0.825	0.00573	0.00473	0.138
$\ll 1$	335	–	0.426	0.816	0.00543	0.00443	0.131

4 Results

We performed simulations at three different turbulent Péclet numbers $Pe_\ell = u\ell/\kappa$ (based on turbulent velocity and length scales u and ℓ): 52, 0.34 and $\ll 1$ (using the SPN approximation). Table 1 shows the values of various physical parameters, including the turbulent diffusion coefficient.

Focusing on the two latter simulations, one can see that the two values of the product of the Richardson and Péclet numbers are very close to each other with a relative difference of 1.4%. This proves that for Péclet numbers smaller than 0.34, the stationary regime of turbulence is characterized by the critical turbulent "Richardson-Péclet" number $(RiPe_\ell)_{\rm cr}$ which is independent of the Péclet number.

Besides, since the two latter values of the β number are also very similar (relative difference of 5.3%), we can write the vertical turbulent diffusion coefficient in the form:

$$D_{\rm t} = \beta\kappa Ri^{-1}(RiPe_\ell)_{\rm cr} \tag{4.1}$$

with a constant β in the $Pe_\ell \leq 0.34$ domain.

Comparing Zahn's prescription (1.1) with our newly derived expression (4.1), the most obvious is that we recover the fact that $D_{\rm t}$ is proportional to κRi^{-1}. Moreover, our simulations provide a first quantitative determination of the proportionality coefficient $\beta(RiPe_\ell)_{\rm cr} = 0.0558$ which is not far from the order of magnitude estimate proposed by Zahn (1992) $Ri_{\rm cr}/3 \simeq 0.0833$.

5 Conclusions

We have validated Zahn's prescription for the vertical turbulent diffusion coefficient in the small-Péclet-number regime, including its dependence on thermal diffusivity, but all considered simulations have been performed with a fixed Reynolds number. The dependence of β and $(RiPe_\ell)_{\rm cr}$ on viscosity is still unexplored.

A future improvement could be to take into account the feedback of the concentration on the flow through molecular weight stratification in order to constrain more complex models (see for example Maeder & Meynet 1996; Talon & Zahn 1997).

Other linked problems would be interesting, like comparing the vertical diffusion coefficient due to a vertical shear (our $D_{\rm t}$, called $D_{\rm v}$ in Zahn 1992) to the horizontal diffusion coefficient $D_{\rm h}$ due to an horizontal shear or varying the angle between shear and stratification. A more complete determination of the dependence of $D_{\rm t}$ on thermal diffusivity is also in progress, with a particular focus on the parameter domain between $Pe_\ell = 52$ and the asymptotic regime.

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