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# A MODEL FOR THE FLUX-R.M.S. CORRELATION IN BLAZAR VARIABILITY OR THE MINIJETS-IN-A-JET STATISTICAL MODEL

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**Abstract.** Very high energy gamma-ray variability of blazar emission remains of puzzling origin. Fast flux variations down to the minute time scale, as observed with H.E.S.S. during flares of the blazar PKS 2155-304, suggests that variability originates from the jet, where Doppler boosting can be invoked to relax causal constraints on the size of the emission region. The observation of log-normality in the flux distributions should rule out additive processes, such as those resulting from uncorrelated multiple-zone emission models, and favour an origin of the variability from multiplicative processes not unlike those observed in a broad class of accreting systems.

We show, using a simple kinematic model, that Doppler boosting of randomly oriented emitting regions generates flux distributions following a Pareto law, that the linear flux-r.m.s. relation found for a single zone holds for a large number of emitting regions, and that the skewed distribution of the total flux is close to a log-normal, despite arising from an additive process.

Keywords: Relativistic processes, Galaxies: jets, Galaxies: active, Gamma rays: bursts

## 1 Introduction

The outbursts of PKS 2155-304 observed at very high energy (VHE, $E>$100 GeV) $\gamma$-rays with H.E.S.S. in July 2006 (Aharonian et al. 2007) constitute "the most dramatic [event] seen from any TeV $\gamma$-ray source" (Longair 2010) and this data-set indeed is a fantastic laboratory for the physics of blazars. Blazars are the Active Galactic Nuclei (AGN) with jets closely aligned with the line-of-sight (see e.g. Urry & Padovani 1995), they may be the best objects to probe this collimated emission of AGN, probably powered by accretion onto the central super-massive black hole (SMBH). Blazars are the prominent class of extragalactic sources detected at high energy (HE, 100 MeV$<E<$100 GeV) and VHE $\gamma$-rays. During scarce periods of high-state emission (so called "flares"), the VHE non-thermal emission can exhibit puzzling properties, the most striking of them probably being hyper-variability. The latter is characterized by an apparent violation of causality, i.e. a flux varying faster than the time needed for information to travel across the emitting region (light crossing time). For example, during the dramatic outbursts of PKS 2155-304 in July 2006, H.E.S.S. observed significant variations down to three *minutes* when the minimum size of the emitting region, bounded by the Schwarzschild radius of the SMBH, is estimated to be at least of three light-*hours*.

The commonly accepted way out this apparent violation of causality is to invoke the relativistic Doppler effect, which induces an observed variation-rate faster by a factor $\delta$, the Doppler factor[i], than the variation time-scale in the emitting region frame. Doppler factors has large as 60 have been invoked to resolve the *minute-hour* discrepancy and to explain the optical thinness of the region to its own radiation (Begelman et al. 2008). The Doppler factor is related to the Lorentz boost and to the orientation of the emitting region, called *minijet* in the following, compared to the line-of-sight. An aligned region would then result in a maximal emission, a "flare", an the misalignment would result in sharp drop of the flux.

Several authors (Ghisellini & Tavecchio 2008; Giannios et al. 2009; Narayan & Piran 2012) have studied the flux resulting from the emission of multiple minijets, without however investigating its statistical properties. During the tremendous high-state of PKS 2155-304, its flux indeed exhibited an highly skewed distribution, interpreted in H.E.S.S. Collaboration, Abramowski et al. (2010) as log-normal[ii] and a linear correlation between the sample[iii] flux and its r.m.s. was observed.

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[i] The derivation of the Doppler factor is extensively discussed in the following.
[ii] If the logarithm of a variable is normally distributed then this variable has a log-normal distribution.
[iii] "Sample" refers here to quantities computed in successive time-windows within the light-curve.

## 2 Statistical properties of the flux of the minijets-in-a-jet

The textbook derivation of the Doppler factor $\delta$ is based on the transformation of velocities in special relativity. We hereafter exploit the fact that $\delta$ is the ratio of the observed to emitted photon energies, where the initial energy is expressed in the frame of isotropic emission. The Doppler effect does not explicitly depend on the energy of an hypothetical photon, but the use of this proxy largely simplifies the problem especially for geometries such as shown in Fig. 1, where a minijet is randomly oriented in a jet non-aligned with the line of sight.

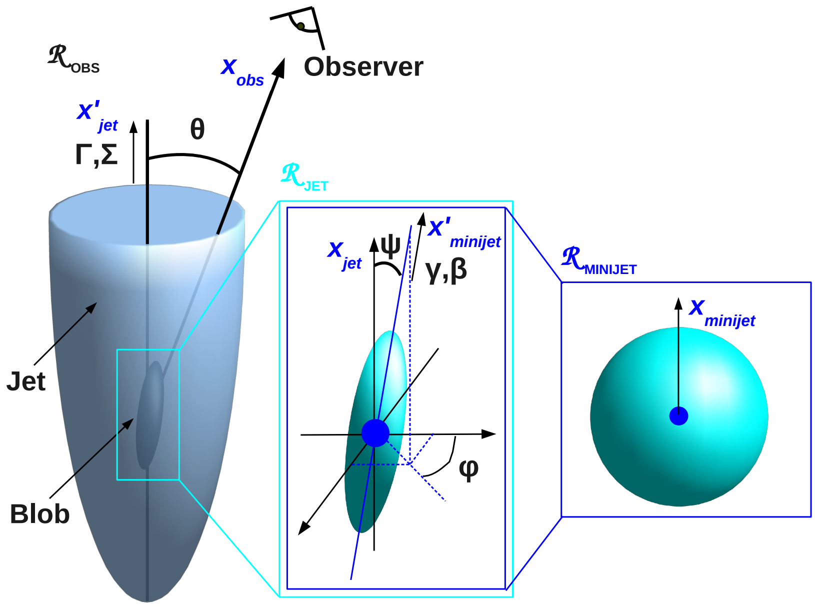


**Fig. 1.** Schematic view of the minijet scenario geometry. In the observer frame $\mathcal{R}_{\rm obs}$, the jet is tilted by angle $\theta$ from $x_{\rm obs}$ and is boosted by a Lorentz factor $\Gamma$ (velocity $\Sigma$) along $x'_{\rm jet}$. The minijet is defined by its Lorentz factor $\gamma$ (velocity $\beta$), in the jet frame $\mathcal{R}_{\rm jet}$. The minijet orientation $x'_{\rm minijet}$ is defined by the spherical angles $\psi$ and $\varphi$, for an isotropic emission in the minijet frame $\mathcal{R}_{\rm minijet}$.

The transformation of the four-momentum, where only photons with energy $E$ travelling along the line of sight are considered ($p_y = p_z = 0$ and $p_x = E$), is given in Eq. 2.1 :

$$\begin{bmatrix} E \\ E \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & {\rm c}_\theta & {\rm s}_\theta & \\ & -{\rm s}_\theta & {\rm c}_\theta & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \Gamma & \Gamma\Sigma & & \\ \Gamma\Sigma & \Gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & {\rm c}_\varphi & {\rm s}_\varphi \\ & & -{\rm s}_\varphi & {\rm c}_\varphi \end{bmatrix} \begin{bmatrix} 1 & & & \\ & {\rm c}_\psi & {\rm s}_\psi & \\ & -{\rm s}_\psi & {\rm c}_\psi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} E_{\rm minijet} \\ p_{x\ \rm minijet} \\ p_{y\ \rm minijet} \\ p_{z\ \rm minijet} \end{bmatrix} \quad (2.1)$$

where c and s are the cosine and sine functions.

Inverting this relation, the time-like component of the equation reads $E_{\rm minijet} = \delta^{-1} E$, with the inverse of the Doppler factor $\delta^{-1} = \gamma\Gamma(1 + \Sigma\beta{\rm c}_\psi - (\Sigma + \beta{\rm c}_\psi){\rm c}_\theta) + \gamma\beta{\rm s}_\theta{\rm s}_\psi{\rm c}_\varphi$. In the blazar case, corresponding to a jet closely aligned with the line-of-sight, $\theta \sim 0$, the extrema of the Doppler factor in the ultra-relativistic limit are achieved for $\delta \geq \Gamma/\gamma$ and $\delta \leq 4\Gamma\gamma$.

Calling $I(E) = E{\rm d}N/{\rm d}E$ the flux intensity at the energy $E$, the quantity $I(E)/E^3$ is a Lorentz invariant (see, e.g., Rybicki & Lightman 1979). Note that the natural energy dependence of a non-thermal spectrum, such as observed at VHE, is a power-law model $I(E) \propto E^{-s}$, where $s$ is the spectral index. Then, the Doppler effect impacts the intensity by a factor $\delta^{3+s}$, which for $\theta = 0$ and $\mu = {\rm c}_\psi$ reads :

$$I(E) \propto \left[\gamma\Gamma(1-\Sigma)(1-\beta\mu)\right]^{-3-s} E^{-s} \equiv (4\Gamma\gamma)^{3+s} g(\mu) E^{-s} \quad (2.2)$$

where $g(\mu) = \left[(1+\Sigma)(1+\beta)/4 \ \times \ (1-\beta)/(1-\beta\mu)\right]^{3+s} \leq 1$ and with $\mu$ uniformly distributed in $[-1, 1]$, assuming an isotropic distribution of the minijets in the jet frame. The distribution of the intensity normalized to its maximum, $I_N = g(\mu)$, can easily be derived using the relation between its probability density function (PDF), $f_\mathcal{I}(I_N)$, and the PDF of $\mu$, $f_\mathcal{C}(\mu) = 1/2$, which, by conservation of the cumulative distribution function under a change of variable, reads $f_\mathcal{I}(I_N) = \left|\partial g^{-1}(I_N)/\partial I\right| f_\mathcal{C}\left(g^{-1}(I_N)\right)$.

The PDF of the normalized distribution is then $f_{\mathcal{I}}(I_N) = (1+\Sigma)/2\beta \times \left[4\gamma^2(3+s)\right]^{-1} \times I_N^{-1-1/(3+s)}$. This power-law dependence of index $1+\alpha$, where $\alpha = 1/(3+s)$, is characteristic of Pareto distributions. Such variables are well known by geophysicists or economists (see the examples given in Newman 2005) and exhibit statistical properties of uttermost interest in the astrophysical case studied here.

First, these variables share a common feature with log-normal variables: they can be seen as the exponential of an underlying variable. Indeed, if a random variable $X$ follows an exponential distribution $f_X(x) = \exp(-\alpha x)$, then $Y = \exp(X)$ is Paretian. Let's assume a small variation $\mathrm{d}x$ of $X$ around $x$, then the resulting variation $\mathrm{d}y$ of $Y$ around $y = h(x)$, where $h$ is the exponential function, simply reads $(\mathrm{d}y)^2 = h'(x)^2(\mathrm{d}x)^2 = \exp(x)^2(\mathrm{d}x)^2 = y^2(\mathrm{d}x)^2$. Herein, $y$ represents the sample flux of the source while $\mathrm{d}y$ is proportional to its r.m.s., so that the previous equation is directly the linear relation between the r.m.s. and its flux.

Secondly, ideal (i.e. un-bounded) Pareto distributions do not follow the central limit theorem (CLT). The CLT indeed relies on the hypothesis that the summed variables admit finite first and second order moments, which are undefined for Pareto distributions with $\alpha \leq 1$ and $\alpha \leq 2$ respectively. A generalized CLT can be applied in such situations and states that the sum of Pareto variables, herein the sum of the emission of minijets, tends to a maximally skewed $\alpha$-stable distribution (see e.g. Zaliapin et al. 2005; Voit 2005).

As shown in the following, such heavy tailed distributions can easily be mistaken for log-normal distributions, with the limited dynamic range inherent to observations. Moreover, the tail of an $\alpha$-stable distribution is a power-law of index $1+\alpha$, so that the linearity between the flux and its r.m.s. is conserved when adding the contributions of a large number of components.

## 3 Simulation of N minijets-in-a-jet

To generate smooth flux-distributions from Eq. 2.2, a simulation with $10^8$ iterations, called hereafter time-steps, for a number of minijets $N \in \{1, 10, 30, 10^2, 3\times10^2, 10^3, 3\times10^3, 10^4\}$ is performed assuming Lorentz factors of the jet $\Gamma = 5$ and of the minijets $\gamma = 5$ (though the numerical values given in the following depend on these parameters, the qualitative results, which are the interest of this study, do not).

The distributions of the logarithm of the minijets summed flux intensities are shown in Fig. 2. For $N \geq 10^3$, the distributions exhibit a peak followed by a power-law tail of index $1+\alpha$. The right-side histograms in Fig. 2 shows the last two distributions, obtained for $N = 3\times10^3$ and $N = 10^4$ minijets, as an observer could see them. With a limited dynamic range (confusion of the low flux bins) and limited statistics or time-coverage (inability to record a significant amount of high flux points), these distributions exhibit similarities with a log-normal process, as shown by the solid black curves.

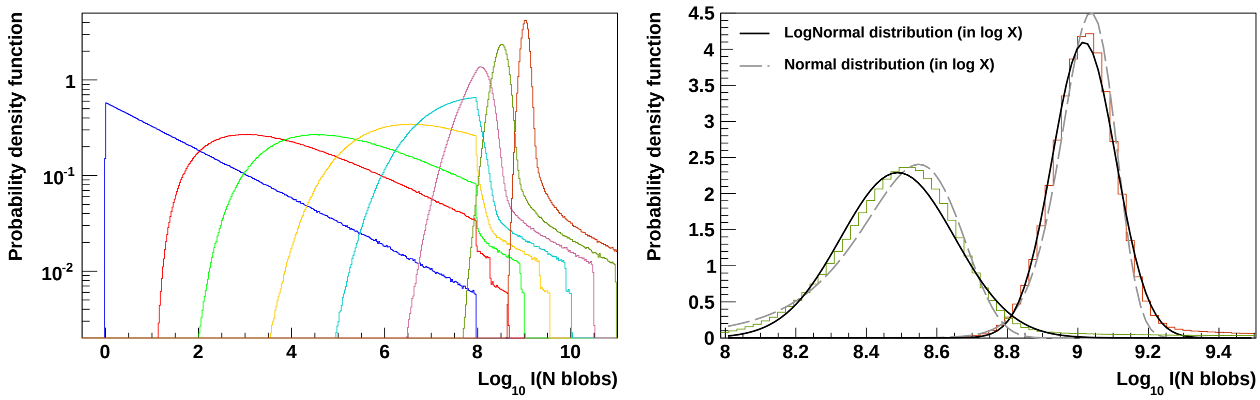


**Fig. 2. Left:** Distribution of the logarithm of the intensity of $N$ independent and randomly oriented minijets. The number of minijets $N$ increases from left to right with $N \in \{1, 10, 30, 10^2, 3\times10^2, 10^3, 3\times10^3, 10^4\}$. Even for a large number of regions, asymmetrical, tailed distributions are obtained. **Right:** Distribution of the logarithm of the flux of $N$ minijets for $N = 3\times10^3$ (left) and $N = 10^4$ (right). The continuous black and grey dashed lines represent the best-fit with a log-normal and normal flux distributions, respectively. Note the linear $y$-axis.

The relation between the flux and its r.m.s. is shown for $N = 1$ and $N = 10^4$ minijets in Fig. 3. Using a simulation of $10^5$ time-steps, the sample flux and its r.m.s. are computed in 10-points wide windows and are grouped in 50 bins of flux for clarity. The uncertainties on the r.m.s. are derived from the variance in each bin.

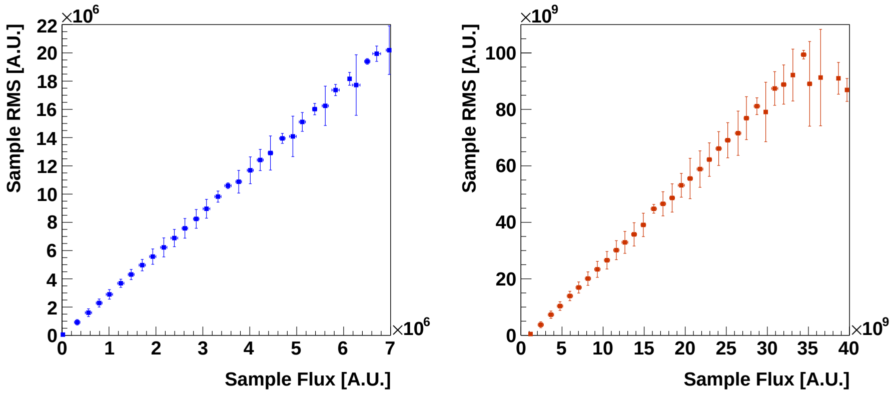


**Fig. 3.** Sample RMS as a function of the sample flux of $N = 1$ minijet (**left**) and the sum of $N = 10^4$ minijets (**right**). Linear relations are found in both cases, with a zero $x$-intercept in the first case and a positive one in the second.

While for $N = 1$, corresponding to a power-law distribution, a strict proportionality between the flux and its r.m.s. is achieved, a positive x-intercept can be seem for $N = 10^4$. This intercept corresponds to the peak of the emission shown in Fig. 2, i.e. to the flux for which the emission of the single components pile-up.

## 4 Conclusion

The dramatic outbursts of PKS 2155-304 constitute an extraordinary set of observables that are highly constraining for models of blazar variability. The observed statistical properties of the emission do not necessarily advocate for a multiplicative process, as could be inferred from a log-normal distribution (see e.g. Uttley et al. 2005; McHardy 2008, in the broader context of accreting objects) and these observables can be reproduced within an additive model summing Pareto variables.

We show that such a Pareto distribution of the flux of a single component is a natural consequence of minijets-in-a-jet modellings, originally developed to explain the hyper-variability, i.e. the apparent violation of causality observed in VHE light-curves. In addition to the skewed distribution of the flux, the sums of Pareto variables follow the linear flux-r.m.s. relation and are then a good representation of both temporal and statistical properties of the emission.

The potential evolutions of this model are diverse, since it can in principle be extended to any astrophysical sources where several boosted regions are involved. Remarkably, Clausen-Brown & Lyutikov (2012) recently derived a particular case ($\Gamma = 1$) of this minijets-in-a-jet statistical model to explain the flares of the Crab in the high energy domain. For this study, we did not focus on the properties of the emission in the Fourier space or as a function of energy, which would require a temporal and an energy-dependent prescriptions. Such developments will certainly be led in future studies.

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