SF2A 2013
L. Cambrésy, F. Martins, E. Nuss and A. Palacios (eds)

LOCAL DIRECT NUMERICAL SIMULATIONS OF TURBULENT MIXING IN STELLAR RADIATIVE ZONES

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Abstract. A major issue of stellar evolution theory is the influence of the transport processes related to rotationally driven macroscopic motions on the internal structure and the evolution of stars. Turbulent mixing of chemical elements due to differential rotation in stellar radiative zones is currently taken into account in many stellar evolution codes through transport coefficients firstly derived by Zahn (1992, A&A, 265, 115). Our aim is to constrain one of these coefficients, the radial diffusion coefficient driven by radial differential rotation through local direct numerical simulations of steady homogeneous stably stratified sheared turbulence, and to compare the results with phenomenological models. In particular, we have determined the dependence of the turbulent diffusion coefficient on thermal diffusivity and chemical stratification.

Keywords: DNS, mixing, stellar interiors, turbulence

1 Introduction

The effects of macroscopic, rotationally induced motions on stellar evolution are still poorly understood. There are large-scale axisymmetric motions, such as differential rotation and meridional circulation, which can be either directly computed in 2D stellar evolution codes or modelled in 1D codes (e.g. the shellular model of Zahn 1992), but there are also small-scale turbulent motions generated by various instabilities. As fundamentally 3D motions, the latter need to be modelled in both 1D and 2D stellar evolution codes. In many current stellar evolution codes, transport of chemical elements is taken into account through a set of diffusion coefficients initially proposed by Zahn (1992).

One of these coefficient, the radial transport coefficient due to shear instability forced by radial differential rotation, is likely to be dominant in radiative zones of stars and is given in Zahn's model by

$$D_{\rm t} = \frac{\kappa}{3}\frac{Ri_{\rm c}}{N^2}\left(r\sin\theta\frac{{\rm d}\Omega}{{\rm d}r}\right) = \frac{\kappa}{3}\frac{Ri_{\rm c}}{Ri}, \tag{1.1}$$

where κ denotes thermal diffusivity; $Ri = (N/S)^2$ the Richardson number, comparing stratification (with the Brunt-Väisälä frequency N) and shear (with the shear rate $S = r\sin\theta{\rm d}\Omega/{\rm d}r$); $Ri_{\rm c}$ the critical value of the latter; r, θ the spherical coordinates and Ω the angular velocity of the star. The derivation of this coefficient is based on four arguments: (i) turbulent flows tend to reach a steady state which is marginally stable; (ii) for high thermal diffusivities, the stability of shear flows is determined by $RiPe$ where $Pe = SL^2/\kappa$ is the Péclet number based on the length scale L of the shear; (iii) the relevant Péclet number in a turbulent state is $Pe_\ell = u\ell/\kappa$, based on turbulent velocity and length scales u and ℓ; (iv) the diffusion coefficient is given by $D_{\rm t} \simeq u\ell/3$.

Models of turbulent transport are currently constrained mainly by measurements of surface chemical abundances from absorption lines (Meynet & Maeder 2000). Recently, internal rotation profiles obtained by asteroseismology allow us to put additional constraints on transport processes (Deheuvels et al. 2012). Here we use local direct numerical simulations of steady homogeneous stably stratified sheared turbulence as numerical experiments to test existing prescriptions.

The framework used to solve our problem is presented in Sect. 2. In Sect. 3 we explore the dependence of the diffusion coefficient on thermal diffusion and in Sect. 4 we study the dynamical effect of chemical stratification on the diffusion coefficient. Finally, we conclude on our results and give some prospects in Sect. 5.

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2 Framework

We study a flow configuration with a uniform vertical shear and vertical uniform temperature and concentration gradients thanks to the versatile (magneto-)hydrodynamical code *Balaïtous*, able to solve either incompressible, Boussinesq or fully compressible equations to study various problems such as convection, dynamo or double diffusion. Its main characteristics are cartesian coordinates, a Fourier collocation method in the horizontal directions, compact finite differences in the vertical one, and a 4th-order Runge-Kutta method in time. We solve the non-dimensional form of the Boussinesq equations with the dynamical effect of chemical stratification:

$$\vec{\nabla}\cdot\vec{v} = 0, \tag{2.1}$$

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p+(Ri\theta+Ri_\mu c')\vec{e}_z+\frac{1}{Re}\Delta\vec{v}, \tag{2.2}$$

$$\frac{\partial\theta}{\partial t}+\vec{v}\cdot\vec{\nabla}\theta+v_z = \frac{1}{Pe}\Delta\theta, \tag{2.3}$$

$$\frac{\partial c'}{\partial t}+\vec{v}\cdot\vec{\nabla}c'+v_z = \frac{1}{Pe_\mathrm{c}}\Delta c', \tag{2.4}$$

where $\vec{v}$ and p are the velocity and the pressure, θ and c' the temperature and concentration fluctuations around the linear mean profiles, Ri_μ the chemical equivalent of the Richardson number, $Re = UL/\nu$ the Reynolds number and $Pe_\mathrm{c} = UL/D_\mathrm{m}$ the chemical Péclet number characterising respectively the viscosity ν and the molecular diffusivity D_m. The small-Péclet-number approximation (SPNA) is an asymptotic development of the Boussinesq equations for very high thermal diffusivities (Lignières 1999), which allow us, with $\psi = \theta/Pe$, to replace Eqs. (2.2) and (2.3) by

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p+(RiPe\psi+Ri_\mu c')\vec{e}_z+\frac{1}{Re}\Delta\vec{v}, \tag{2.5}$$

$$v_z = \Delta\psi. \tag{2.6}$$

The first of Zahn's hypotheses is verified by tuning Ri (respectively $RiPe$ in SPNA simulations) so that we reach a statistical steady state. Then, we determine the turbulent diffusion coefficient thanks to the relation

$$D_\mathrm{t} = -\frac{\langle v_z c'\rangle}{\mathrm{d}C/\mathrm{d}z}, \tag{2.7}$$

where $\langle\rangle$ denotes the spatial average and $C(z)$ the mean concentration profile.

3 Thermal diffusion

We are interested here in the case $Ri_\mu = 0$. As described in Prat & Lignières (2013), we have performed a series of Boussinesq simulations at decreasing Péclet numbers down to 0.34 and one simulation in the SPNA, which show that the quantity $D/(\kappa Ri^{-1})$ tends to a constant value as Pe_ℓ decreases, as illustrated in Fig. 1. Thus, D_t is proportional to κRi^{-1}, which is in agreement with Zahn's model (1.1). In addition, we are able to give a quantitative estimate of the proportionality constant, around 5.58×10^{-2} *vs.* 8.33×10^{-2} in Zahn (1992), and of the critical value $Ri_\mathrm{c} = 0.427$, what is quite different from the classical value 1/4.

Figure 1 also clearly shows that Zahn's model is not valid out of the domain of small Péclet numbers. We performed other simulations with increasing Péclet numbers up to $Pe_\ell = 437$ and found that in this regime, our results are incompatible with the model proposed by Maeder (1995) but in good agreement with that of Lindborg & Brethouwer (2008), proposed in the geophysical literature, where the diffusion coefficient reads

$$D_\mathrm{t} = \frac{\varepsilon_\mathrm{P}}{N^2}, \tag{3.1}$$

ε_P denoting the specific dissipation rate of turbulent potential energy. Figure 2 represents the ratios between each model (Zahn's, Lindborg's and the sum) and the diffusion coefficient estimated in our simulations. We see that the sum of both models is close to the estimated coefficient for the whole range of Péclet numbers.

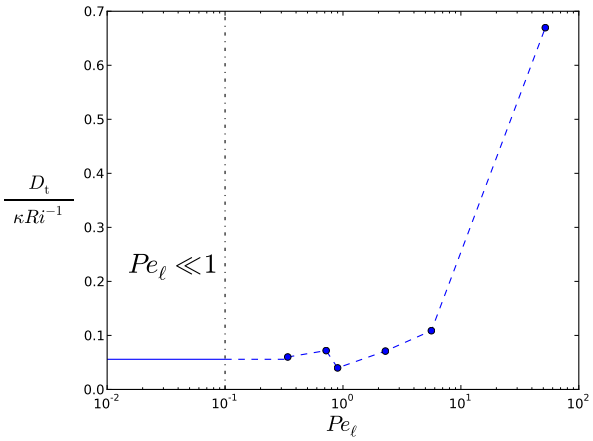

Fig. 1. $D_t/(\kappa Ri^{-1})$ as a function of Pe_ℓ. Dots correspond to Boussinesq simulations, the solid line to the SPNA one.

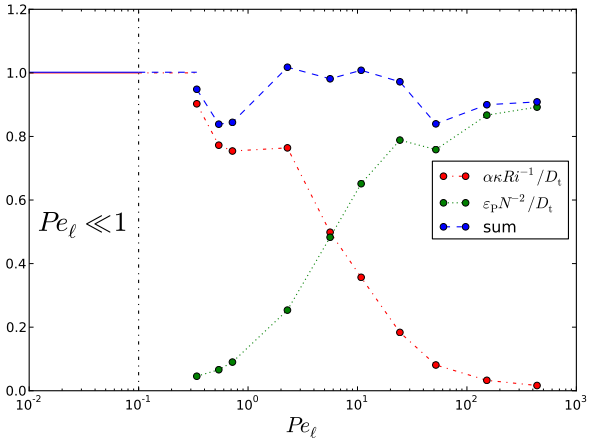

Fig. 2. Ratios between models and effective diffusion coefficient as a function of the turbulent Péclet number

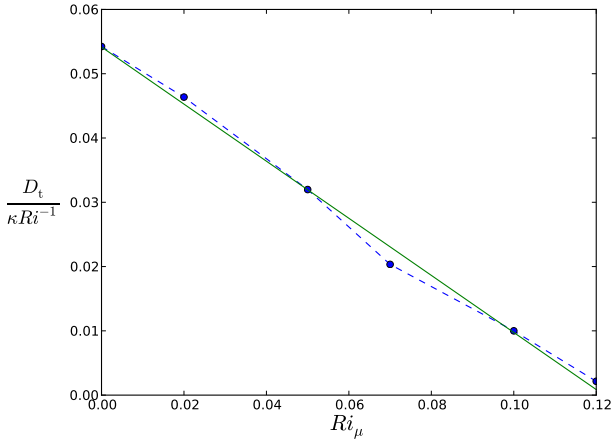

Fig. 3. $D_t/(\kappa Ri^{-1})$ as a function of Ri_μ. Dots correspond to simulations and the solid line to the linear regression.

4 Chemical stratification

We performed simulations in the SPNA with different values of the chemical Richardson number Ri_μ and estimated for each one the value of the turbulent diffusion coefficient. As shown in Fig. 3, the quantity $D_\mathrm{t}/(\kappa Ri^{-1})$ is well represented by an affine function of Ri_μ. We can thus write

$$\frac{D_\mathrm{t}}{\kappa Ri^{-1}} = \beta(Ri_\mathrm{c} - Ri_\mu), \tag{4.1}$$

with $\beta = 0.45$ and $Ri_\mathrm{c} = 0.12$. Then, it yields

$$D_\mathrm{t} = \beta\kappa\frac{Ri_\mathrm{c} - Ri_\mu}{Ri}. \tag{4.2}$$

Equation (4.2) has been derived by Maeder & Meynet (1996) in the domain of small Péclet numbers using the hypothesis that the stabilising effect of chemical stratification is not affected by thermal diffusion and can thus completely inhibit the transport. This is not the case of all other models used in stellar evolution theory.

The model of Maeder (1997), for example, is based on the hypothesis that thermal diffusion affects equally thermal and chemical stratifications. As we can see in Eq. (2.5), this is incompatible with the SPNA, where κ and Ri can be very large while Ri_μ remains finite. In this limit, the diffusion coefficient of Maeder (1997),

$$D_\mathrm{t} = \beta\kappa\frac{Ri_\mathrm{c}}{Ri + Ri_\mu}, \tag{4.3}$$

reduces to a form similar to Eq. (1.1), which does not depend on Ri_μ. This is clearly not what we observe in Fig. 3. This model is thus incompatible with our numerical simulations in the domain of small Péclet numbers.

The case of Talon & Zahn (1997) is more complex, as they assume that a very efficient horizontal diffusion is able to reduce the stabilising effect of chemical stratification. In our simulations, horizontal diffusion is not much larger as the vertical one, so we cannot conclude on the validity of this model.

5 Conclusion

This work demonstrates the potential of numerical simulations applied to the physics of stellar interiors. Indeed, in the domain of small Péclet numbers, we have validated Zahn's model in steady homogeneous stably stratified sheared simulations. In addition, we are able to give a quantitative estimate of the turbulent diffusion coefficient. In the domain of large Péclet numbers, we have shown that the diffusion coefficient is given by the model of Lindborg & Brethouwer (2008), which is incompatible with the model of Maeder (1995). As for the dynamical effect of chemical stratification, our simulations have allowed us to choose between two existing models, Maeder & Meynet (1996) and Maeder (1997). A recent study by Meynet et al. (2013) compares the predictions of a stellar evolution code using various prescriptions for radial mixing and shows that the latter model has the best fit with the observations, thus suggesting that a significant additional source of mixing may exist.

To go further, compare to other models and possibly propose new prescriptions, several features need to be implemented, such as the effect of horizontal diffusion induced by horizontal differential rotation. Another idea is to study the influence of a magnetic field on shear instability, as initiated by Lecoanet et al. (2010).

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