SF2A 2013
L. Cambrésy, F. Martins, E. Nuss and A. Palacios (eds)

RELATIVISTIC MODELS FOR GAIA AT THE (CROSS)CHECK-POINT

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Abstract. Given the extreme accuracy of modern space astrometry, a precise relativistic modeling of observations is required. Concerning light propagation, most approaches rely on the solution of the null-geodesic equations. However, another approach based on the Time Transfer Functions (TTF) can be used to define an astrometric observation using an integral-based method derived from the Synge World Function. The availability of several models, formulated in different and independent ways, is indeed a security against the presence of systematic errors in the analysis of future experimental results. It is the case of the forthcoming Gaia mission. In this work, we review the modeling of a Gaia-like astrometric observation using the TTF and two other approaches, namely the Gaia RElativistic Model (GREM) and the Relativistic Astrometric MODel (RAMOD), and we provide explicit relations between their characteristic quantities.

Keywords: astrometry, General Relativity, Gaia, Time Transfer Functions, light propagation

1 Introduction

The space astrometry mission Gaia (Bienayme & Turon 2002), planned to be launched by the European Space Agency (ESA) at the end of this year, will observe about one billion objects in the magnitude range from 6 to 20. In particular, the astrometric core solution will determine the astrometric parameters (position, parallax, and proper motion) for a subset of these sources with an accuracy of several μas. It is an extremely difficult task but also a crucial one for the outcome of the mission. This solution will be performed by the Astrometric Global Iterative Solution (AGIS) software (Lindegren et al. 2012). At the same time, an independent verification unit for AGIS called Global Sphere Reconstruction (GSR) (Vecchiato et al. 2012) has been set within the Gaia Data Processing and Analysis Consortium (DPAC). Both pipelines are intended to operate on the same real data and the comparison of their results will validate the final astrometric catalog. In order to keep the two software as separate as possible, two different relativistic modelings of light propagation have been implemented: AGIS relies on GREM (Klioner 2003), while GSR implements RAMOD (de Felice et al. 2006). Moreover, an independent approach to the modeling of the astrometric observables based on the TTF (Teyssandier & Le Poncin-Lafitte 2008) has been recently put in place. We review these three approaches by translating them into the same notation and providing explicit relations between their characteristic quantities used to describe light propagation in a curved space-time.

2 Notations

In this paper c is the speed of light in a vacuum and G is the Newtonian gravitational constant. The Lorentzian metric of space-time V_4 is denoted by g. The signature adopted for g is $(-+++)$. We suppose that space-time is covered by a global quasi-Galilean coordinate system $(x^\mu) = (x^0, \boldsymbol{x})$, where $x^0 = ct$, t being a time coordinate, and $\boldsymbol{x} = (x^i)$. We assume that $g_{00} < 0$ anywhere. We employ the vector notation $\boldsymbol{a}$ in order to denote $(a^1, a^2, a^3) = (a^i)$. Considering two such quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ we use $\boldsymbol{a} \cdot \boldsymbol{b}$ to denote $a^i b^i$ (Einstein convention on repeated indices is used). The quantity $a = |\boldsymbol{a}|$ stands for the ordinary Euclidean norm of $\boldsymbol{a}$. For any quantity $f(x^\lambda)$, $f_{,\alpha}$ and $\partial_\alpha f$ denote the partial derivative of f with respect to x^α. The indices in

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parentheses characterize the order of perturbation. They are set up or down, depending on the convenience. In the following, we will consider that space-time is perturbed by a system of gravitational mass monopoles represented by a weak-field metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. It has been showed in (Klioner 2003) that in order to get the μas angular accuracy required by modern space astrometry, the gravitational perturbation shall be expanded in the parametrized post-Newtonian (PPN) approximation of GR as

$$h_{00} = \frac{2G}{c^2} \sum_P \frac{\mathcal{M}_P}{R_P(t,\boldsymbol{x})} \quad , \quad h_{0i} = 0 \quad , \quad h_{ij} = \delta_{ij} \gamma h_{00} \; , \tag{2.1}$$

with $\mathcal{M}_P$ the mass of the perturbing body P and $\boldsymbol{R_P}(t,\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}_P(t_C)$, where $\boldsymbol{x}_P(t_C)$ is the position of P at the time of maximum approach with the photon and γ is a PPN parameter (Will 1993).

3 Astrometric observables in the Time Transfer Functions formalism

The goal of astrometry is to determine the position of celestial bodies from angular measurements. One way to get a covariant definition of the astrometric observable is to use the tetrad formalism (Weinberg 1972; Misner et al. 1973; Brumberg 1991), thus giving the direction of observation of an incoming light ray in a particular frame comoving with the observer.

Let us note $\lambda^{\mu}_{(\alpha)}$ the components of this tetrad, where (α) corresponds to the tetrad index and μ is a normal tensor index which can be lowered and raised by using the metric. It has been shown in (Bertone & Le Poncin-Lafitte 2012) that we can express the direction of the light ray in the tetrad frame as

$$n^{(i)} = -\frac{\lambda^0_{(i)} + \lambda^j_{(i)} \hat{k}_j}{\lambda^0_{(0)} + \lambda^j_{(0)} \hat{k}_j} = -\frac{\lambda^0_{(i)} + \hat{k}_j \lambda^j_{(i)}}{u^0 \left(1 + \hat{k}_j \beta^j\right)} \; , \tag{3.1}$$

where $\widehat{k}_i = k_i/k_0$ is the so called light direction triple with $k_\mu = g_{\mu\nu} k^\nu$ the covariant components of the tangent vectors to the light ray $k^\mu \equiv dx^\mu/d\lambda$, u^α represents the unit four-velocity of the satellite and $\beta^i \equiv v^i/c$, v^i being the coordinate velocity of the observer.

Let us suppose the existence of a unique light ray connecting the emission event of the signal $x_A = (ct_A, \boldsymbol{x}_A)$ and its reception event $x_B = (ct_B, \boldsymbol{x}_B)$. In addition, we put $R^i_{AB} \equiv x^i_B - x^i_A$. Under these hypothesis, in the TTF formalism one defines the light direction triple within the post-Minkowskian (PM) approximation of General Relativity (GR) as closed form integrals of the metric tensor and its derivatives along the Minkowskian straight line $z^\alpha_-(\lambda) = (ct_B - R_{AB}, \boldsymbol{x}_B - \boldsymbol{R_{AB}})$ as (Bertone et al. 2013; Hees et al. 2013)

$$\left(\widehat{k}_i\right)_B \equiv \left(\frac{k_i}{k_0}\right)_B = N^i_{AB} - \frac{1}{2} \int_0^1 \left[R^i_{AB} \lambda m_{,0} - R_{AB}(1-\lambda) m_{,i} - \tilde{h}_i\right]_{z_-(\lambda)} d\lambda + \mathcal{O}(h^2), \tag{3.2}$$

where we define $N^i_{AB} = R^i_{AB}/R_{AB}$ and

$$m_{,\alpha} \equiv h_{00,\alpha} + 2N^k_{AB} h_{0k,\alpha} + N^j_{AB} N^k_{AB} h_{jk,\alpha} \; , \tag{3.3a}$$

$$\tilde{h}_i \equiv N^i_{AB} h_{00} - N^i_{AB} N^j_{AB} N^k_{AB} h_{jk} + 2h_{0i} + 2N^j_{AB} h_{ij} \; . \tag{3.3b}$$

Taking the metric (2.1), Eq. (3.2) can be written as

$$(\widehat{k}^B_i)(\boldsymbol{x}_A, \boldsymbol{x}_B) = N^i_{AB} - (\gamma+1) \sum_P \frac{G\mathcal{M}_P}{c^2 R_{PB}} \frac{1}{1 + \boldsymbol{N}_{PA} \cdot \boldsymbol{N}_{PB}} \times \left[\frac{R_{AB}}{R_{PA}} \boldsymbol{N}_{PB} - \left(1 + \frac{R_{PB}}{R_{PA}}\right) \boldsymbol{N}_{AB}\right] \; , \tag{3.4}$$

where we use $\boldsymbol{R}_{PX} = \boldsymbol{x}_X - \boldsymbol{x}_P(t_C)$. Then, a generic tetrad comoving with the chosen observer and computed at the same accuracy can be used in Eq. (3.1) along with Eq. (3.4) to compute the direction of light in the observer reference frame.

4 The Gaia astrometric models

In the context of the Gaia mission two independent relativistic models have been developed to analyze and interpret the observations: GREM (Klioner 2003) and RAMOD (de Felice et al. 2004). In this section, we shall detail the definition of an astrometric observable in these two models.

4.1 GREM

GREM is actually the most complete relativistic model for astrometry and the basis for the reduction of Gaia observations. it sets several steps for the conversion of the observed quantities into the coordinate ones, from the observed direction of light to the spatial position of the emitter in the BCRS (Klioner 2003).

Adopting the IAU resolution B1.3 (Soffel et al. 2003) for the BCRS and the Comoving Reference System (ComRS), (Klioner 2003) gives the explicit coordinate transformation between the aberration-free direction n^i and the observed direction s^i at μas accuracy as

$$\begin{aligned} s^i &= -n^i + c^{-1}\Big[\boldsymbol{n}\times(\boldsymbol{v_s}\times\boldsymbol{n})\Big]^i \\ &+c^{-2}\left\{(\boldsymbol{n}\cdot\boldsymbol{v_s})\Big[\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v_s})\Big]^i+\frac{1}{2}\Big[\boldsymbol{v_s}\times(\boldsymbol{n}\times\boldsymbol{v_s})\Big]^i\right\} \\ &+c^{-3}\left\{\Big[(\boldsymbol{v_s}\cdot\boldsymbol{n})^2+(1+\gamma)w(\boldsymbol{x})\Big]\Big[\boldsymbol{n}\times(\boldsymbol{v_s}\times\boldsymbol{n})\Big]^i+\frac{1}{2}(\boldsymbol{v_s}\cdot\boldsymbol{n})\Big[\boldsymbol{v_s}\times(\boldsymbol{n}\times\boldsymbol{v_s})\Big]^i\right\}+\mathcal{O}(c^{-4})\ , \end{aligned} \tag{4.1}$$

where $\boldsymbol{v}_s$ is the coordinate velocity of the observing satellite and $w(\boldsymbol{x})$ is the PPN gravitational potential. Eq. (4.1) contains the aberrational effects up to $1/c^{-3}$. The "aberration free direction" is given by the unit vector $\boldsymbol{n}\equiv\boldsymbol{p}/|\boldsymbol{p}|$, with

$$p^i \equiv c^{-1}\dot{x}^i = \sigma^i + c^{-1}\Delta\dot{x}^i(t)\ , \tag{4.2}$$

with $\dot{x}^i$ the coordinate direction of the photon. To compute Eq. (4.2) at reception coordinates, we need then the unperturbed direction of the light ray at past null infinity

$$\sigma^i = N^i_{AB} - \frac{2G}{c^2}\sum_P \frac{\mathcal{M}_P}{R_{AB}}\left[\frac{(\boldsymbol{N}_{AB}\times\boldsymbol{R}_{PB}\times\boldsymbol{N_{AB}})^i}{R^2_{PB}-(\boldsymbol{N_{AB}}\cdot\boldsymbol{R}_{PB})^2}\big(R_{PA}-R_{PB}-R_{AB}\big)\right]+\mathcal{O}(c^{-4}) \tag{4.3}$$

and the gravitational deflection of the light ray $\Delta\dot{x}^i(t_B)$ given by

$$\frac{\Delta\dot{x}^i_B}{c} = -\frac{2G}{c^2}\sum_P \mathcal{M}_P\frac{1}{R_{PB}}\left\{N^i_{AB}+\frac{(\boldsymbol{N_{AB}}\times\boldsymbol{R_{PA}}\times\boldsymbol{N_{AB}})^i}{R_{PB}-\boldsymbol{N_{AB}}\cdot\boldsymbol{R_{PB}}}\right\}\ . \tag{4.4}$$

Finally, some classical transformations to subtract the parallax and proper motion effects allow to get the barycentric direction of the source.

4.2 RAMOD

Developed by an italian group of relativists and astronomers, RAMOD (de Felice et al. 2004) is based on a complete PM background and it always relies on measurable quantities with respect to a local barycentric observer along the light ray. The unknown is then the local line-of-sight, representing the direction of the light ray in the rest space of the observer $\boldsymbol{u}$, and defined as

$$\bar{\ell}^\alpha = -\frac{k^\alpha}{u_\beta k^\beta} - u^\alpha\ . \tag{4.5}$$

In this formalism, the null-geodesic equation transforms, according to the measurement protocol procedure, into a set of coupled nonlinear differential equations, called "master equations" (de Felice et al. 2004) whose solution at reception coordinates x_B in the case of metric (2.1) gives (Crosta 2013; Bertone et al. 2013)

$$\bar{\ell}^k_B = \frac{x^k_B - x^k_A}{R_{AB}} + \frac{2G}{c^2}\sum_P \mathcal{M}_P\left\{-\frac{N^k_{AB}}{2R_{PB}}+\frac{d^k_B}{d^2_B}\left[-\frac{\boldsymbol{N_{AB}}\cdot\boldsymbol{R_{PB}}}{R_{PB}}+\frac{R_{PB}-R_{PA}}{R_{AB}}\right]\right\}+O\left(c^{-4}\right)\ , \tag{4.6}$$

where $d^i_B \equiv R^i_{PB} - N^i_{AB}(\boldsymbol{R_{PB}}\cdot\boldsymbol{N_{AB}})$. Once solved for $\bar{\ell}^k$, the astrometric observable is defined by its projection on a tetrad $E^\beta_{\hat{a}}$ comoving with the observer (Bini et al. 2003) as

$$n_{(i)} = \frac{(\bar{\ell}_B - \nu)_\beta E^\beta_{(i)}}{\bar{\gamma}(1-\nu_\alpha\bar{\ell}^\alpha_B)}\ , \tag{4.7}$$

where ν_α is the four-velocity of the satellite u_s relatively to the local barycentric observer u and $\bar{\gamma} = -u^\alpha_s u_\alpha$.

An explicit expression of the tetrad $E^\beta_{\hat{a}}$ adapted to Gaia has been obtained in (Crosta & Vecchiato 2010) by successive transformations of the local BCRS tetrad (Bini et al. 2003). Using this approach, RAMOD provides a complete modeling of the astrometric observable.

5 Setting explicit relations among the astrometric models

The comparison of GREM and RAMOD is considered as a priority for the Gaia mission (Crosta & Vecchiato 2010; Crosta 2011) since they will both concur to the creation of the mission catalog. Any inconsistency in the relativistic models would then invalidate the quality and reliability of the estimates, hence all related scientific output. Both models are internally consistent at the μas level required by Gaia but their different conception makes an analytical cross-check a complex task.

In a recent work by (Crosta & Vecchiato 2010), the authors show that the PN expansion of Eq. (4.7) up to the order needed at Gaia accuracy yields the equivalence with the observable defined in Eq. (4.1) for GREM. However, a complete analytical cross-check of the modeling for gravitational light deflection has not been done yet. Moreover, despite their differences, both models aim for the reconstruction of light trajectory from the star to the observer in order to build the astrometric observable. This makes the TTF model presented in section 3 a completely new and independent approach to the problem, particularly adapted to contribute to the verification process of the relativistic solutions to be used for the Gaia mission.

In this section, we present the relations allowing to cross-check the relativistic description of light propagation in GREM, RAMOD and our TTF model by computing the light direction triple $\widehat{k}_i$ in the three models, then by comparing the resulting formulae in the gravitational field of point bodies.

5.1 GREM

The relation between the light direction triple and the photon velocity $\dot{x}^i$ used in GREM is given by

$$\hat{k}_i = \frac{g_{ij}k^j + g_{0i}k^0}{g_{00}k^0 + g_{0i}k^i} = -\frac{\dot{x}^i}{c} - 2h_{00}\sigma^i + \mathcal{O}(c^{-4}) , \tag{5.1}$$

noting that $\dfrac{\dot{x}^i}{c} = \dfrac{dx^i/d\lambda}{dx^0/d\lambda} = \dfrac{k^i}{k^0}$. Using Eq. (4.3), Eq. (4.4) and the metric tensor (2.1) into Eq. (5.1) taken at the reception event, it is then straightforward to check the equivalence with Eq. (3.4).

5.2 RAMOD

The relation between the light direction triple $\widehat{k}_i$ and the local line-of-sight $\bar{\ell}^i$ of RAMOD at reception point x_B is obtained by expanding Eq. (4.5) at the order c^{-3} required for Gaia. Considering that $u^0 = 1 + \dfrac{h_{00}}{2} + O(c^{-4})$, we get

$$\left(\widehat{k}_i\right)_B = -\bar{\ell}^i_B \left[1 + \frac{3}{2}h_{00}\Big|_B\right] + \mathcal{O}(c^{-3}) . \tag{5.2}$$

Substituting for $\bar{\ell}^i_B$ from Eq. (4.6) and using the metric given in Eq. (2.1) into Eq. (5.2), the reader can easily retrieve Eq. (3.4). Combining this result with Eq. (4.7), one also gets the equivalence with the TTF astrometric observable (3.1).

6 Conclusions

In this communication we review three relativistic approaches to the modeling of an astrometric observation, namely GREM, RAMOD and the one based on the TTF. We describe the steps allowing to get the unperturbed direction of the source of a light signal from the direction measured by an observer in motion within a gravitational field. To apply our analysis in the context of the Gaia mission, we work in the weak-field approximation of GR, considering a metric tensor adapted to the μas accuracy. Finally, we give explicit relations between the quantities defined in each approach to describe the direction of a light ray at its reception coordinates. Using these relations (5.1)-(5.2) it is straightforward to show the equivalence of the three approaches at the μas level of accuracy. We shall note that such a cross-checking procedure enters the same thread of model comparison started in (Crosta & Vecchiato 2010), which is essential to fully understand the observational data coming from Gaia in a common experimental context.

S. Bertone thanks UIF/UFI (French-Italian University) for the financial support of this work. S. Bertone, C. Le Poncin-Lafitte and M.-C. Angonin are grateful to the financial support of CNRS/GRAM and CNES/Gaia. The work of M. Crosta and A. Vecchiato has been partially funded by ASI under contract to INAF I/058/10/0 (Gaia Mission - The Italian Participation to DPAC).

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