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RESONANCES DUE TO THIRD BODY PERTURBATIONS IN THE DYNAMICS OF MEOS

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Abstract. The dynamics of Medium Earth Orbits (MEO) sees nowadays a renewed interest because of the development of satellites constellations (GNSS), that raises the problem of parking orbits for satellites at end of life. Numerical evidence shows that the resonances related to the presence of a third body can affect the stability of MEO orbits. The goal of our work is to study the effects of resonances on the stability of MEO orbits, over long or very long term (hundreds of years). For orbits above 20,000 km altitude, the perturbation due to a third body (the Moon or the Sun) is not fully negligible, so we take into account the third body perturbation on the secular evolution of the angular variable of the satellite orbits, and thus of the resonant angle. We estimate numerically the evolution of the resonant inclination. We study the stability of some resonances analytically and numerically, and in particular the resonance associated to the operational inclination of the Galileo satellites.

Keywords: Medium Earth Orbits, resonances, stability.

1 Introduction

The region of space hosting the Medium Earth Orbits (MEO) is becoming more and more crowded, especially because of the increasing presence of the satellites constellation, like the GPS and GALILEO families. The lifetime of these satellites is limited, and they have to be replaced regularly. This rises the problem of positioning the inactive satellites on "safe" parking orbits, in order to avoid the classical problems with spatial debris: they can become a danger for the active satellites and can cause the multiplication of the debris by collision. For this reason, understanding the stability properties of MEO orbits for long times can guide the choice of suitable parking orbits. Among the various factors, the study of this stability includes an analysis of the behaviour of the orbits in presence of resonances, which can be a possible source of instability or, on the contrary, can help to keep the orbit stable.

The long term effects of the resonances on the stability of MEO orbits is not yet well understood. Moreover, in the litterature there are only few articles about the case of MEO orbits. In Rossi (2008) several numerical experiments concerning the interactions between resonances and MEO orbits are presented. For the theory of the resonances, the main references are the two papers Hughes (1980) and Hughes (1981), mainly dedicated to list the resonant angular combinations. Some analytical studies about the resonances dependent only on the eccentricity can be found in Breiter (1999) and Breiter (2001), but only for the case of low earth orbits.

In this paper, we deal with resonances due to the presence of a third body, that can be either the Sun or the Moon. These resonances involve linear combinations of the angles of the satellites and of the third body which appear in the development of the potential due to this third body. Such combinations write

$$\psi = (n-2p)\omega + m\Omega + (n-2p+q)M + (n-2k)\omega_{3\mathrm{b}} + m\Omega_{3\mathrm{b}} + (n-2k+j)M_{3\mathrm{b}} \tag{1.1}$$

where ω, Ω, l are the argument of the perigee, the longitude of the ascending node and the mean anomaly of the satellite and $\omega_{3\mathrm{b}}$, $\Omega_{3\mathrm{b}}$, $M_{3\mathrm{b}}$ are the same angles associated to the third body. A resonance occurs when

$$\dot{\psi} \approx 0\,.$$

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While the frequencies associated to the third bodies (Sun and Moon) are well fixed, the frequencies associated to the satellite strongly depend of its orbit. Hughes (Hughes (1980) and Hughes (1981)) has carefully studied the relation between the possible resonances and the characteristics of the orbit. He has in particular evidenced a class of resonances related to combinations of the form $\alpha g + \beta h$ which involve only secular variations of the satellite angles. These variations are mainly due to the Earth's oblateness (related to the J_2 coefficient of the geopotential) and in this case, the resonance condition is only function of the inclination. One of these resonances occurs for an inclination of about 56° which is very close of the inclination of the Galileo orbits. The study of this particular resonance is thus of practical interest. This paper is a first attempt in this direction.

The article is organized as follows. In section 2, we set up the model for the hamiltonian, including the Earth's gravity potential reduced to its monopolar and quadrupolar terms and the third body gravity potential; moreover we average the hamiltonian with respect to every non resonant angles in order to keep only the driven terms for the very long term evolution (over tens of years). Section 3 contains the results of the study of this averaged system for the case of one selected resonance.

2 The model of resonance

Our fundamental variables will be the Delaunay canonical variables l, g, h, L, G, H but we will also use for convenience the classical orbital elements $a, e, i, \Omega, \omega, M$. The two sets of variables are related by the relations

$$l = M, \quad g = \omega, \quad h = \Omega,$$
$$L = \sqrt{\mu a}, \quad G = L\sqrt{1-e^2}, \quad H = G\cos i\,.$$

Our model takes into account the main part of the Earth's gravity potential as well as the perturbation of a third body, i.e. the Sun or the Moon, which is the source of the resonances, the main subject of this paper. The Hamiltonian of the system is decomposed as

$$\mathcal{J} = \mathcal{J}_{\mathrm{Kep}} + \mathcal{J}_{J_2} + \mathcal{J}_{\mathrm{3b}}\,; \tag{2.1}$$

$\mathcal{J}_{\mathrm{Kep}}$ is the Hamiltonian corresponding to the keplerian motion, $\mathcal{J}_{J_2}$ is the perturbation due the Earth's oblateness and $\mathcal{J}_{\mathrm{3b}}$ is the perturbation due to the third body. The full expression of these well known perturbations can be found, for example, in Stefanelli et al. (2012). In particular, in this paper, we will focus on the Sun perturbation (so $\mathrm{3b} = \odot$).

Since we want to study the orbital evolution over very long time spans (typically several tens of years), we will discard the short periodic terms related to the fast variables l and $M_\odot$. This is achieved by means of a classical canonical (time dependent) change of variables not detailed here, so that the new hamiltonian no longer depends on l and $M_\odot$.

It is well known that in the J_2 problem, the elimination of the mean anomaly is accompanied, at the first order of the transformation, by the elimination of the perigee argument g (Brouwer (1959)). Thus, we obtain the new Hamiltonian (in principle, we should use new (primed) variables but we will keep the same notations for the sake of simplicity)

$$\overline{\mathcal{J}}_{J_2} = \frac{1}{4} J_2 \frac{\mu}{a}\left(\frac{R_{\mathrm{e}}}{a}\right)^2 \frac{1}{\eta^3}\left(1 - 3\cos^2 i\right)\,, \tag{2.2}$$

where $\mu = Gm_e$, and R_e are the Earth's gravitational constant and equatorial radius, respectively, J_2 is the coefficient of the second order zonal armoniccs of the Earth's gravitational potential and $\eta = \sqrt{1-e^2}$.

Concerning the Sun perturbation involving linear combinations of the angles, removing l and $M_\odot$ is equivalent to select the appropriate angular combinations (Stefanelli et al. (2012)):

$$\overline{\mathcal{J}}_\odot = -\mu_\odot \frac{a^2}{a_\odot^3}\mathcal{H}_{2,1,0}(e_\odot)\sum_{m=0}^{2}(2-\delta_{0m})\frac{(2-m)!}{(2+m)!}\sum_{p=0}^{2}\bar{\mathcal{F}}_{2,m,p}(i)\bar{\mathcal{F}}_{2,m,1}(\epsilon)\mathcal{G}_{2,p,2p-2}(e)\cos\left((2-2p)g + mh\right)\,, \tag{2.3}$$

where ϵ is the obliquity, $\bar{\mathcal{F}}_{n,m,p}$ denotes the modified Allan inclination functions (Allan (1965)) and $\mathcal{G}_{n,p,q}$, $\mathcal{H}_{n,p,q}$ are the eccentricity special functions related to the classical Hansen coefficients (Kaula (1966)).

Note that $\overline{\mathcal{J}}_\odot$ does no longer depend on $\Omega_\odot, \omega_\odot, M_\odot$ and thus on time.

Now the averaged Hamiltonian takes the form

$$\overline{\mathcal{J}} = \overline{\mathcal{J}}_{\mathrm{Kep}}(-,-,-,L,-,-) + \overline{\mathcal{J}}_{J_2}(-,-,-,L,G,H) + \overline{\mathcal{J}}_\odot(-,g,h,L,G,H)\,. \tag{2.4}$$

In this paper we will deal with the resonances due to the presence of the Sun. Potentially, a large class of resonances of the form $\alpha\dot{g} + \beta\dot{h} \approx 0$ exists (Stefanelli et al. (2012), Hughes (1980)) but we focus on the resonance, that we will call 2:1,

$$2\dot{g} + \dot{h} \approx 0\,. \tag{2.5}$$

Indeed, this resonance has one of the largest amplitude and it affects orbits with inclination close to 56° or 110°, concerning in particular the Galileo orbits.

To this end, we perform a suitable canonical change of variable $(g, h, G, H) \to (\sigma, \xi, \Sigma, \Xi)$ so that the resonant angle $\sigma = 2g + h$ becomes one of the new angles; one possible choice is the following:

$$\begin{aligned} &\sigma = 2g + h\,, \quad \Sigma = H\,,\\ &\xi = g\,, \quad \Xi = G - 2H\,. \end{aligned} \tag{2.6}$$

In terms of these new variables, the phase entering in $\overline{\mathcal{J}}_\odot$ becomes:

$$(2 - 2p)g + mh = m\sigma + 2(1 - p - m)\xi\,.$$

The next step consists in eliminating the non resonant angles from the Hamiltonian. In other words, we select in the sum (2.3) only terms either free of angular variables or containing only the resonant angle σ:

$$\overline{\overline{\mathcal{J}}}_\odot = -\mu_\odot \frac{a^2}{a_\odot^3}\mathcal{H}_{2,1,0}(e_\odot)\Big[\bar{\mathcal{F}}_{2,0,1}(i)\bar{\mathcal{F}}_{2,0,1}(\epsilon)\mathcal{G}_{2,1,0}(e) + \bar{\mathcal{F}}_{2,1,0}(i)\bar{\mathcal{F}}_{2,1,1}(\epsilon)\mathcal{G}_{2,0,-2}(e)\cos(\sigma)\Big]\,. \tag{2.7}$$

In the general case, the new Hamiltonian can be decomposed as

$$\overline{\overline{\mathcal{J}}} = \overline{\overline{\mathcal{J}}}(\sigma, \Sigma, \Xi; L) = \overline{\overline{\mathcal{J}}}_{\mathrm{Kep}} + \overline{\overline{\mathcal{J}}}_{J_2,\mathrm{sec}} + \overline{\overline{\mathcal{J}}}_{\odot,\mathrm{sec}} + \overline{\overline{\mathcal{J}}}_{\odot,\mathrm{res}}\,; \tag{2.8}$$

where $\overline{\overline{\mathcal{J}}}_{\mathrm{Kep}}(L) = \overline{\mathcal{J}}_{\mathrm{Kep}} = \mathcal{J}_{\mathrm{Kep}}$ and $\overline{\overline{\mathcal{J}}}_{J_2,\mathrm{sec}}(\Sigma, \Xi; L) = \overline{\mathcal{J}}_{J_2}$; we have split the perturbation due to the Sun into a secular part $\overline{\overline{\mathcal{J}}}_{\odot,\mathrm{res}}(\Sigma, \Xi; L)$, free of angular variables, and a resonant part $\overline{\overline{\mathcal{J}}}_{\odot,\mathrm{res}}(\sigma, \Sigma, \Xi; L)$.

3 Study of the resonance 2:1

The main idea of this section is to study the properties of a resonance by means of the study of the equilibrium points of the Hamiltonian system in the new canonical variables $(\sigma, \xi, \Sigma, \Xi)$.

After performing the averaging and the change of coordinates described in the previous section, we will now study the new Hamiltonian, labelled $\mathcal{K}$, function of the new canonical variables $(\sigma, \xi, \Sigma, \Xi)$, given by

$$\mathcal{K} = \mathcal{K}(\sigma, \xi, \Sigma, \Xi) = \mathcal{K}_{\mathrm{Kep}} + \mathcal{K}_{J_2,\mathrm{sec}} + \mathcal{K}_{\odot,\mathrm{sec}} + \mathcal{K}_{\odot,\mathrm{res}}\,. \tag{3.1}$$

The equations of motion associated to this Hamiltonian system are:

$$\dot{\sigma} = \frac{\partial\mathcal{K}}{\partial\Sigma}\,, \quad \dot{\Sigma} = -\frac{\partial\mathcal{K}}{\partial\sigma}\,, \tag{3.2}$$

$$\dot{\xi} = \frac{\partial\mathcal{K}}{\partial\Xi}\,, \quad \dot{\Xi} = -\frac{\partial\mathcal{K}}{\partial\xi}\,. \tag{3.3}$$

Since the Hamiltonian does not depend on ξ, the action Ξ is constant and the value and time evolution of ξ are irrelevant, we take into account only the equations (3.2) that, in view of (3.1), are now:

$$\dot{\sigma} = \frac{\partial\mathcal{K}_{J_2}}{\partial\Sigma} + \frac{\partial\mathcal{K}_{\odot,\mathrm{sec}}}{\partial\Sigma} + \frac{\partial\mathcal{K}_{\odot,\mathrm{res}}}{\partial\Sigma}\,, \qquad \dot{\Sigma} = -\frac{\partial\mathcal{K}_{\odot,\mathrm{res}}}{\partial\sigma}\,, \tag{3.4}$$

with Ξ becoming a parameter set by the initial conditions.

It can be shown that the system (3.4) takes the form

$$\dot{\sigma} = A(a, e, i) + B(a, e, i)\cos\sigma\,, \qquad \dot{\Sigma} = e^2 C(a, e, i)\sin\sigma\,. \tag{3.5}$$

In particular, A is such that $A(a, e, i) = f(a, e)P(\cos i; a, e)$, where $P(\cos i; a, e)$ is a polynomial of degree 2 in $\cos i$ (with coefficients dependent on eccentricity and semimajor axis) admitting two roots $i_1^* \approx 56$ and $i_2^* \approx 110$ for each fixed couple of parameters a, e.

Looking for the equilibrium points means to solve

$$\dot{\sigma} = 0, \qquad \dot{\Sigma} = 0\,; \tag{3.6}$$

note that in oder to have $\dot{\Sigma} = 0$ we have three possibilities.

- $e = 0$: since $B \sim O(e)$, the equation $\dot{\sigma} = 0$ reduces to

$$A(a, 0, i) = 0$$

that admits the two aforementioned solutions (or resonant inclinations) i_1^* and i_2^*. Note that these are equilibrium points regardeless to the value of σ.

- If $\sin\sigma = 0$, or equivalently $\sigma = 0$ or π, the first equation reduces to

$$A(a, e, i) \pm B(a, e, i) = 0\,; \tag{3.7}$$

for fixed a and e, this gives resonant inclinations i_1^{**} and i_2^{**}. Since the amplitude of the resonant Sun perturbation (related to the function B) is small, i_1^{**} and i_2^{**} are just slightly displaced w.r.t. the resonant inclinations i_1^* and i_2^* of the previus case. To resume, we have four equilibrium points,

$$\sigma = 0 \text{ or } \pi \qquad \text{and} \qquad i = i_1^{**} \text{ or } \imath_2^{**}$$

and we can show numerically, plotting the phase portraits (figures 1 and 2), that the points such that $\sigma^* = 0$ are two stable points (centers) and $\sigma^* = \pi$ are two unstable points (seddles).

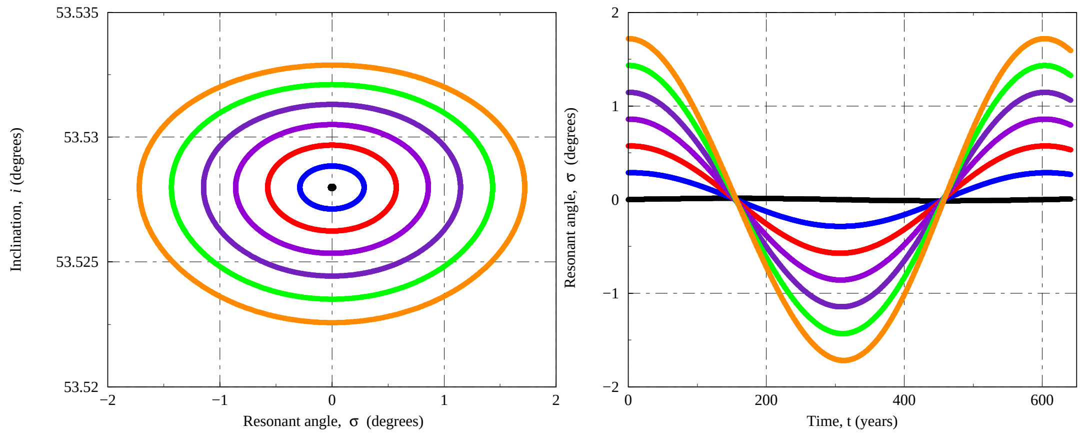

Fig. 1. Initial eccentricicty $e_0 = 0.1$. **Left:** Phase portrait in a neighborhood of one of the centres, $(\sigma, i) = (0, i_1^{**})$. **Right:** Time evolution of the resonant angle for seven orbits with initial conditions in a right neighbourhood of the centre point.

- Finally, if $i = 0$ or π we have $C(a, e, i) = 0$ and the first equation is satisfied if

$$\sigma = \pm \arccos \frac{A(a, e, i = 0)}{B(a, e, i = 0)}$$

and the same for $i = \pi$. The stability of this class of equilibrium point will be investigated in a future work.

4 Conclusions

The aim of this work is to understand the effects of lunisolar resonances on the stability of Medium Earth Orbits over long periods (of the order of more centuries). The source of resonance is the presence of a third massive body, which can be the Sun or the Moon; this kind of resonance is due to combinations of frequencies of the satellite and of the third body.

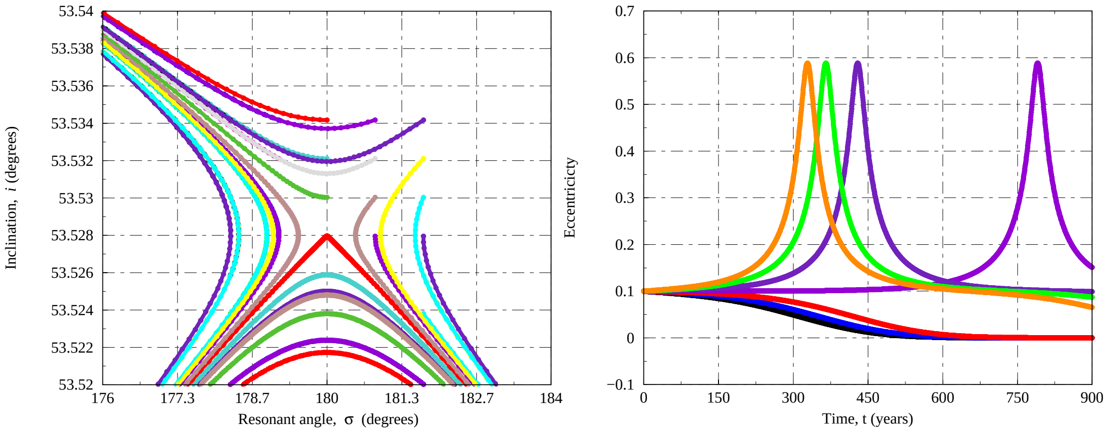

Fig. 2. Initial eccentricicty $e_0 = 0.1$. **Left:** Phase portrait in a neighborhood of the seddle point $(\sigma, i) = (\pi, i_1^{**})$. **Right:** Time evolution of the eccentricity for seven orbits with initial conditions in a neighbourhood of the seddle point.

On the other hand, the orbit of the satellite, and thus its frequencies, is affected by many perturbations: mainly from the Earth's gravity potential, but also, at intermediate altitudes, by the potential of the third body itself; in this work we considered the Sun. Thus, beside the Earth oblateness, our model includes the secular terms of the third body potential and a resonant perturbation related to a selected resonance, $2\omega + \Omega$. This seems to be among the most important lunisolar resonances and the associated resonant inclination is very close to the operational inclinations of the Galileo orbits.

We combined an analytical approach (especially in the case taking into account only the J_2 secular effect and the case of circular orbits) with direct numerical integrations.

The analytical approach, making use of an Hamiltonian model, allowed to study in detail this resonance, looking at the (linearized) equilibrium points of the averaged systems. This method is quite general and could be also extended to other resonances in the future. This allowed to charachterize the stabilty of equilibrium points for the simplest case (only J_2 secular effect) and ensure that, when adding the Sun perturbation, the equilibrium points are just slightly displaced. The numerical integration of the model states that the equilibrium points preserve their stability character and allows to study the time evolution of the other orbital elements of resonant orbits; in particular, the perspective is to focus on the behaviour of the eccentricity, which is of fundamental importance to ensure that orbital crossings do not occur.

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