SF2A 2007
J. Bouvier, A. Chalabaev, C. Charbonnel (eds)

CLOSED-LOOP ADAPTIVE OPTICS PERFORMANCE WITH THE FRACTAL ITERATIVE METHOD

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Abstract. We demonstrate the interest of using the Fractal Iterative Method as a fast algorithm for the Internal Model Control of closed-loop adaptive optics systems. Such a method reduces the noise propagation in comparison with the classical Least-Squares reconstructors, and only a few iterations are required to maintain a well-balanced error budget of the system. Furthermore, the internal model control might yield interesting robustness and stability properties, still under characterization.

1 Introduction

Initially Adaptive Optics (AO) systems in closed-loop were only expected to cancel residual measurements, that is to correct on-axis perturbations, with an integrator for the control law (Gendron & Lena 1994). Nowadays, the various concepts of AO systems under study for the future telescopes can no longer be solely based on this objective. It has been demonstrated (Béchet et al. 2006) that the general optimization of the Strehl ratio can lead to an Internal Model Control (IMC) law. In the challenging case of large AO systems for the Extremely Large Telescopes, this architecture can be implemented in practice thanks to the Fractal Iterative Method (FrIM) (Thiébaut & Tallon 2007). This combination complies with requirements on both correction quality and fast computation. First, we present the equations of our system with an IMC. Then, we analyse the evolution of FrIM algorithm convergence rate depending on the signal-to-noise ratio.

2 Internal Model Control for a Closed-Loop System

For the new concepts of AO systems, such as multi-conjugate AO or multi-object AO, the criterion to optimize must be formulated in the turbulent layers planes, so as to flatten the wavefront in several layers or in each observing direction. Such a correction ensures the best image quality, assessed by the Strehl ratio. The block-diagram of the equivalent discrete-time system is represented by Fig. 1.

The command vector $\mathbf{a}_k$ of the k-th AO loop is obtained (Béchet et al. 2006) by computing

$$\begin{aligned}\mathbf{a}_k &= \mathbf{F}\cdot\mathbf{E}\cdot(\mathbf{d}_k+\mathbf{G}\cdot\mathbf{a}_{k-1}) = \mathbf{M}^\dagger\cdot\mathbf{P}\cdot\mathbf{E}\cdot(\mathbf{d}_k+\mathbf{G}\cdot\mathbf{a}_{k-1})\ , \qquad (2.1)\\ &= \mathbf{M}^\dagger\cdot\mathbf{P}\cdot\mathbf{C}_{\mathbf{w},1}\cdot\mathbf{C}_{\mathbf{w},0}^{-1}\cdot\mathbf{R}^\dagger\cdot(\mathbf{d}_k+\mathbf{G}\cdot\mathbf{a}_{k-1})\ , \qquad (2.2)\end{aligned}$$

where $\mathbf{d}_k$ is the vector of closed-loop data, that are spatial derivatives of the residual wavefront, and $\mathbf{a}_{k-1}$ is the vector of the previous command, currently applied during integration of measurements $\mathbf{d}_k$. Both in Fig. 1 and in Eq. (2.1), the matrix $\mathbf{F}=\mathbf{M}^\dagger\cdot\mathbf{P}$ stands for the control part, with $\mathbf{M}^\dagger$ being the pseudo-inverse of the Deformable Mirror Influence Matrix and $\mathbf{P}$ a linear operator combining a telescope-aperture weighting and a projection onto piston-removed subspace of the wavefront. $\mathbf{E}$ illustrates the estimation part, including the reconstruction of the wavefront thanks to $\mathbf{R}^\dagger$ and some prediction thanks to $(\mathbf{C}_{\mathbf{w},1}\cdot\mathbf{C}_{\mathbf{w},0}^{-1})$. Operator $\mathbf{R}^\dagger$ is the Minimum-Variance reconstructor derived from the open-loop optimization (Béchet et al. 2006; Thiébaut & Tallon 2007)

$$\mathbf{R}^\dagger = (\mathbf{S}^{\mathrm{T}}\cdot\mathbf{C}_{\mathbf{n}}^{-1}\cdot\mathbf{S}+\mathbf{C}_{\mathbf{w},0}^{-1})^{-1}\cdot\mathbf{S}^{\mathrm{T}}\cdot\mathbf{C}_{\mathbf{n}}^{-1} \qquad (2.3)$$

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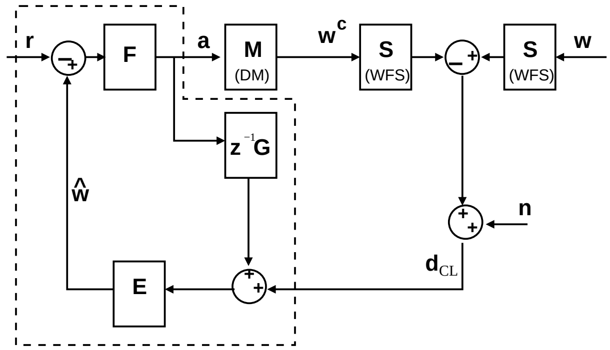

Fig. 1. The diagram of the closed-loop AO system enhances the estimation (**E**) and control (**F**) parts directly derived from the criterion optimization, as well as the internal model constituted by the interaction matrix **G**.

where $\mathbf{S}$ is the linear model of the wavefront sensor, $\mathbf{C_n}$ is the covariance matrix of measurement noise and $\mathbf{C}_{\mathbf{w},0}$ is the spatial covariance of the turbulent wavefront. The rationale to use in Eq. (2.2) the same priors as for an open-loop reconstruction is that the term $(\mathbf{d}_k + \mathbf{G} \cdot \mathbf{a}_{k-1})$ is an estimate of open-loop measurements.

Applying $\mathbf{R}^\dagger$ thus provides an estimate of the observed turbulent wavefront. Prediction in Eq. (2.2) involves the covariance matrix of two successively observed turbulent wavefronts $\mathbf{C}_{\mathbf{w},1} = \langle \mathbf{w}_{k+1} \mathbf{w}_k^{\mathrm{T}} \rangle$. Owing to the very short temporal sampling period, with respect to the evolution of the turbulence, the simplest priors is to consider the turbulent wavefront to be the same from one loop to the next, hence $\mathbf{C}_{\mathbf{w},1} \simeq \mathbf{C}_{\mathbf{w},0}$. Thereafter, in our simulations, the control law is

$$\mathbf{a}_k = \mathbf{M}^\dagger \cdot \mathbf{P} \cdot \mathbf{R}^\dagger \cdot (\mathbf{d}_k + \mathbf{G} \cdot \mathbf{a}_{k-1}) \ . \tag{2.4}$$

When the model **G** of the Interaction Matrix of the system is perfect then the IMC is equivalent to an open-loop control of the AO system (Morari & Zafiriou 1989). This point raises two interesting aspects of the IMC architecture here. On the first hand, with a perfect model of **G**, our performance study would apply for a closed-loop system as well as for an open-loop one. On the second hand, stability problems are only due to the discrepancy between the model **G** for the Interaction Matrix and the real process linking actuators values to measured local derivatives. This is good news because, experimentally, it is easier to have an accurate model for the Interaction Matrix than to separately have good models for the Deformable Mirror (**M**) and for the wavefront sensor (**S**).

3 FrIM performance for large closed-loop AO

Adaptive Optics systems for the next generation of ground-based telescopes will have to cope with a huge number of degrees of freedom N_{Dof}. For $N_{Dof} = 10^4$, this would imply the solving of a linar system with a matrix of dimension $10^4 \times 10^4$, at the frequency of about 1kHz. In such a case, the full inverse matrix storage as well as the matrix-vector multiplication at high frequency, scaling as $O(N_{Dof}^2)$, become serious computational issues. To face these problems, several novel algorithms for wavefront reconstruction have been suggested. One of them, FrIM (Thiébaut & Tallon 2007), has been studied in the IMC design presented above. FrIM has proven quick convergence on simulations (Thiébaut & Tallon 2007; Béchet et al. 2006), independently of the system dimension. The total computational burden scales as $O(N_{Dof})$.

As it is based on an iterative solving of the linear system

$$(\mathbf{S}^{\mathrm{T}} \cdot \mathbf{C}_{\mathbf{n}}^{-1} \cdot \mathbf{S} + \mathbf{C}_{\mathbf{w},0}^{-1}) \cdot \widehat{\mathbf{w}} = \mathbf{S}^{\mathrm{T}} \cdot \mathbf{C}_{\mathbf{n}}^{-1} \cdot (\mathbf{d}_k + \mathbf{G} \cdot \mathbf{a}_{k-1}) \ , \tag{3.1}$$

the convergence speed affects the total number of operations to be computed in an AO loop. It appeared necessary to define a criterion so as to specify the computational burden of FrIM in a closed-loop AO system. This criterion is obtained from the constraint to have a well-balanced error budget for the AO system. The two only terms of the error budget (Le Louarn et al. 1998) concerned by the number of iterations and consequently by the time needed to compute them are the reconstruction error and the temporal error associated to the

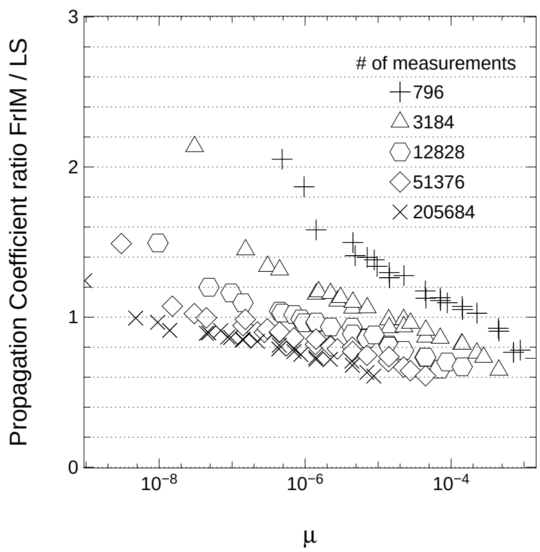

Fig. 2. Ratio between FrIM noise propagation coefficient $\gamma_{\mathbf{n}}^{\mathrm{MV}}$ and LS one $\gamma_{\mathbf{n}}^{\mathrm{LS}}$ versus $\mu = \sigma_{\mathbf{n}}^2/(6.88(D/r_0)^{5/3})$. Plotted symbols correspond to values obtained from simulations. Symbols change with the AO system number of degrees of freedom.

AO delay. The reconstruction error is the mean-square of the difference between the real wavefront and its estimation from (3.1),

$$J_1 = \langle \|\mathbf{P}\cdot(\mathbf{w}_k - \mathbf{R}^{\dagger}\cdot(\mathbf{d}_k + \mathbf{G}\cdot\mathbf{a}_{k-1}))\|^2\rangle \tag{3.2}$$

The temporal error is computed as the mean-square difference between the currently observed wavefront and the one which produced the data in (3.1),

$$J_2 = \langle \|\mathbf{P}\cdot(\mathbf{w}_{k+1} - \mathbf{w}_k)\|^2\rangle. \tag{3.3}$$

A shorter exposure time reduces J_2, but increases J_1, for leading to higher measurement uncertainties. The minimum total error is obtained for $J_1 \simeq J_2$, which corresponds to a well-balanced system.

To simulate well-balanced AO systems, we have generated successive turbulent layers with an adjusted discrepancy such that $J_1 = J_2$. Then we could determine the minimum number of iterations required to preserve quality. In order to apply this constraint, the first point was to study the estimation error provided by the Minimum-Variance reconstructor of FrIM.

Unlike Least-Squares (LS) reconstruction error J_1^{LS}, the Minimum-Variance one, J_1^{MV}, depends on both the noise level and the turbulence strength. The turbulent wavefront covariance is approximated by

$$\mathbf{C}_{\mathbf{w},0} = 6.88(D/r_0)^{5/3}\overline{\mathbf{C}}_{\mathbf{w},0} \simeq \mathbf{K}\cdot\mathbf{K}^{\mathrm{T}} \tag{3.4}$$

where D is the pupil diameter, r_0 is Fried's parameter and $\mathbf{K}$ being the fractal operator (Thiébaut & Tallon 2007) . Written this way, the normalized covariance matrix $\overline{\mathbf{C}}_{\mathbf{w},0}$ only depends on the number of measurements, and no longer on D nor on r_0. In the usual assumption of uniform and uncorrelated measurements uncertainties, the noise covariance writes $\mathbf{C}_{\mathbf{n}} = \sigma_{\mathbf{n}}^2\mathbf{I}$, where $\mathbf{I}$ is the Identity matrix. Then, the noise propagation coefficients of the two reconstructors, defined as the ratio $J_1/\sigma_{\mathbf{n}}^2$ for a uniform measurement noise, are

$$\gamma_{\mathbf{n}}^{\mathrm{LS}} = \frac{J_1^{\mathrm{LS}}}{\sigma_{\mathbf{n}}^2} = \mathrm{tr}\left[\mathbf{P}\cdot\left(\mathbf{S}^{\mathrm{T}}\cdot\mathbf{S}\right)^{-1}\cdot\mathbf{P}^{\mathrm{T}}\right] \quad \text{and} \quad \gamma_{\mathbf{n}}^{\mathrm{MV}} = \frac{J_1^{\mathrm{MV}}}{\sigma_{\mathbf{n}}^2} = \mathrm{tr}\left[\mathbf{P}\cdot\left(\mathbf{S}^{\mathrm{T}}\cdot\mathbf{S} + \mu\,\overline{\mathbf{C}}_{\mathbf{w},0}^{-1}\right)^{-1}\cdot\mathbf{P}^{\mathrm{T}}\right] \tag{3.5}$$

where tr is the Trace operator and the factor μ stands for the ratio $\sigma_{\mathbf{n}}^2/(6.88\,(D/r_0)^{5/3})$.

One must notice that *mu* is inversely proportional to the square of the signal-to-noise ratio. The relative evolution of $\gamma_{\mathbf{n}}$ versus μ, according to Eq. (3.5), is plotted in Fig. 2, with various values for the N_{DoF} of the system. Actually the plots of Fig. 2 represent the ratio between the reconstruction error obtained with FrIM, and the one obtained with the LS reconstructor. For LS, Noll (1978) showed that propagation coefficient was

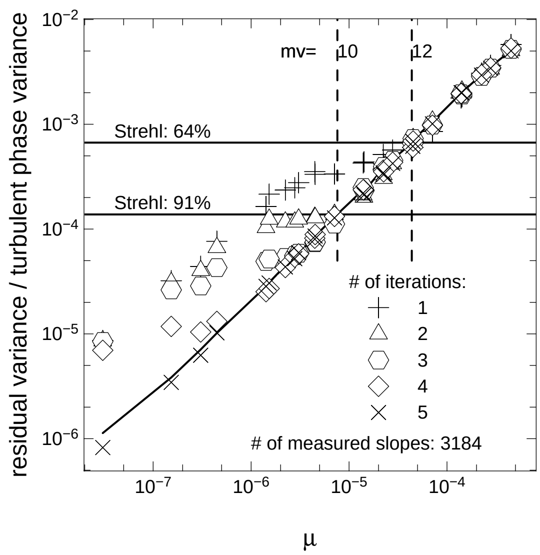

Fig. 3. Evolution of FrIM performance with the number of iterations allowed in a closed-loop AO in the case of 64 subapertures along the pupil diameter. For typical seeing conditions at Paranal, equivalent Strehl ratio and magnitude of Natural Guide Star are indicated with horizontal and vertical lines.

increasing proportionally to $log(N_{DoF})$. However, LS noise propagation coefficient does not depend on the noise level, which is not the case for the Minimum-Variance with FrIM. Given a system dimension, FrIM will perform better than the LS for high values of μ, that is for low signal-to-noise ratio. On the contrary, given a signal-to-noise ratio, FrIM can reach lower noise propagation than the LS when the number of degrees of freedom gets larger.

Having quantified the reconstruction error at convergence, J_1, for every conditions of simulations, *i.e.*for all combination of number of measurements and μ values, it is now possible to close the loop and to simulate a well-balanced AO system. If only 2 iterations can be computed during each AO loop, then we could not always reach a total error close to $2 \times J_1^{\mathrm{MV}}$. The simulations results plotted on Fig. 3 show the ratio between the total residual error and the variance of the turbulent wavefront above the telescope pupil, depending on μ. An interesting way to analyse the results of Fig. 3 is to associate the value of μ with a given Guide Star magnitude and the variance of the residual phase ordinate with the Strehl ratio. For instance, considering the mean conditions at Paranal (Le Louarn et al 1998), with a seeing of 0.7arcsec, a telescope diameter $D = 42$m, a correction wavelength of $\lambda_{\mathrm{AO}} = 2.2\mu$m and 64 subapertures accross the diameter, the horizontal lines in Fig. 3 represent two Strehl ratio thresholds: SR $= 91\%$ and 64%. Assuming an AO loop frequency of 500Hz, a transmission of 0.5 and a 100nm-bandwidth, μ values in abscissae can be linked with the magnitude of the Guide Star. Vertical dashed lines state the magnitudes $m_v = 10$ and 12. One may notice the tradeoff between Guide Star magnitude, reachable Strehl ratio and number of iterations of the algorithm. Typically for a magnitude $m_v = 12$, more than one single iteration does not imprve the Streh ratio, whereas with $m_v = 10$, the possible Strehl ratio of 91% requires at least two iterations to be achieved. In this last case, the computational burden for the algorithm is $(13 + 33 \times N_{iter}) \times N_{DoF} = 79 \times N_{DoF}$ number of operations.

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