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THE SCALING LAWS PROBLEM IN RADIATING HYDRODYNAMICS

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Abstract. In this paper we present a rigorous and systematic method, based on the Lie group theory, which allows to study the similarity properties, to establish the scaling laws and to obtain self-similar solutions. We will show that it is possible to unify different problems through this method. Moreover, we derive, by local considerations, a complete equivalence between laboratory systems and astrophysical objects. We examine, in details, the case of optically thin radiating fluids and consider the example of radiating shock in magnetic cataclysmic variables.

1 Introduction

For a few years astrophysics benefits from High-Energy Density (HED) Physics development (Remington et al. 2006). The main interest is to explore the matter in extreme states of density and temperature which could only be met in astrophysics. The experimental reproduction of astrophysical phenomena allows to validate theoretical models and numerical methods. In order to evaluate the astrophysical character of HED experiments the similarity properties of created plasma must be studied. The establishment of similarity and scaling laws constitutes a milestone of laboratory astrophysics and must be studied through a rigorous formalism. In this paper we consider a theoretical approach based on Lie groups which will be described later on. Our approach is based on a particular one-parameter Lie group called the homothetic Lie group. In the context of laboratory astrophysics several works have been realized in several hydrodynamic regimes reproducible with powerful facilities. Each of them considers global similarity properties using dimensional analysis. Ryutov et al.(1999) and Ryutov et al.(2000) have considered respectively the hydrodynamics and magnetohydrodynamics scaling laws problem. The hydrodynamics studied by Ryutov et al.(1999) includes the Birkhoff polytropic symmetries (Birkhoff 1950). Ryutov et al.(2001) have examined optically thin radiative hydrodymanics scaling laws and Castor (2007) has studied the similarity problem in the case of optically thin plasmas and discussed from a microscopic approach. A recent work (Falize et al. 2007) has been performed to determine scaling laws and similarity properties of optically thick regime [diffusion approximation o(v^2/c^2)]. In the present paper we will deal with similarity problems in hydrodynamics when matter is optically thin. We will establish scaling laws associated to the considered radiating regime. In the first part we discuss the method that is used and remind its fundamental concepts. In the second part, we consider optically thin radiating fluids which are a major topic in astrophysics. We take as an example the case of radiating shocks in cataclysmic variables.

2 Lie groups theory and similarity properties

The invariant transformation group theory elaborated by Sophus Lie is one of the most powerful theories of mathematical physics to study the symmetry properties partial differential equations possess and to perform their analytical integration. Among all the Lie transformation groups, one of them, *i.e.* the one-parameter homothetic group, is widely used due to its simplicity. In the point of view of self-similar solutions, this group provides more solutions than dimensional analysis. This is because the homothetic group is a sub-group of scaling transformations. Considering the fundamental principle of Laboratory Astrophysics (*i.e.* to recreate systems which have astronomical dimensions on short scales), it seems natural to use the homothetic group in

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order to study similarity properties, scaling laws and even self-similarity. Here, we will just consider the first two points.
The invariance of equations by the homothetic group implies that Rankine-Hugoniot relations are also preserved. Thus, we make sure that shocks are reproduced on small scales and constitute homothetic structures of astrophysical shocks. After these fundamental considerations, we present results of the problem we studied.

3 Similarity and scaling laws of optically thin radiating fluid

Several astrophysical systems have a cooling (or heating) characteristic time which gets close to the dynamical time and consequently these radiative phenomena should be taken into account in the modeling. For optically thin plasmas, a simple modeling of radiating losses (or heating) can be done by the mere introduction of a loss (or gain) of entropy. Thus, the equations of the model write:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}[\rho\vec{v}]=0,\quad \rho\frac{d}{dt}\vec{v}=-\vec{\nabla}P,\quad \frac{dP}{dt}-\gamma\frac{P}{\rho}\frac{d\rho}{dt}=-(\gamma-1)\mathcal{L}(\rho,T),\quad dM=\rho(r,t)dV \tag{3.1}$$

where d/dt is the Lagrangian derivative and ρ, P, T, $\vec{v}$, $\mathcal{L}$, γ are respectively the density, the pressure, the temperature, the velocity, the source function and the polytropic index. In addition, we suppose $\mathcal{L}(\rho,T)=\mathcal{Q}_1(\rho,T)+\mathcal{Q}_2(\rho,T)$ where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two source terms. So far, the sources have not been specified , but in the applications we will consider the power laws forms.

Table 1. *Group invariants affiliated to local dimensionless numbers.*

	invariant quantities	Dimensionless numbers
I_1	vr/t	Strouhal (St)
I_2	$Pt/(\rho vr)=I_1P/(\rho v^2)\propto 1/\mathcal{M}^2$	Euler number (Eu) or Mach number ($\mathcal{M}$)
I_3	γ	Polytropic index
I_4	$M/(\rho r^{1+d})$	Mass and geometry conservation
I_5	$P/(\epsilon\rho^{\alpha}T^{\beta})$	Equation of state
I_6	Q_1t/P	t/t_{Q_1} Cooling or heating time
I_7	Q_2t/P	t/t_{Q_2} Cooling or heating time

We must include an equation of state to close the equations system and connect the microscopic phenomena to the cooling function. We suppose an equation of state as power law type: $P=\varepsilon_0(Z)\rho^{\mu}T^{\nu}$ where ϵ_0 is a coefficient depending on Z which is the ionization. To fulfill entropy conservation we must impose the relation $\gamma(1-\nu)=(\mu-\nu)$. In an astrophysical point of view the objects that are concerned by such modelling are interstellar jets and bow shocks, radiating shocks (Point C in the Drake diagram (Drake 2006 fig 7.17) in Polars (Cropper 1990) or supernova remnants (Chevalier & Blondin 1995)). The relation between the astrophysical objects characteristics and the laboratory characteristics (that we top with $\sim$) are given by: $r=a^{\delta_1}\tilde{r}$, $t=a^{\delta_2}\tilde{t}$, $\vec{v}=a^{\delta_3}\tilde{\vec{v}}$, $M=a^{\delta_4}\tilde{M}$, $\rho=a^{\delta_5}\tilde{\rho}$, $P=a^{\delta_6}\tilde{P}$, $\mathcal{Q}_1=a^{\delta_7}\tilde{Q}_1$, $\mathcal{Q}_2=a^{\delta_8}\tilde{Q}_2$, $\varepsilon_0=a^{\delta_9}\tilde{\varepsilon}_0$, $T=a^{\delta_{10}}\tilde{T}$, $\gamma=a^{\delta_{11}}\tilde{\gamma}$, where a is the group parameter and δ_i are the homothetic exponants . The rescaling of the quantities ε_0, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ includes possible modification of ionization. In these calculations we consider unspecified source functions and in the applications we take a power law form ($\mathcal{Q}_1=\mathcal{Q}_{0,1}\rho^{m_1}P^{n_1}r^{l_1}$, $\mathcal{Q}_2=\mathcal{Q}_{0,2}\rho^{m_2}P^{n_2}r^{l_2}$ where $\mathcal{Q}_{0,1}$ and $\mathcal{Q}_{0,2}$ are two constants). These types of source function is completely justified in the case of cooling phenomena. Indeed many processes in the continuum can be modeled by a power laws dependance. We can also write $Q_i\propto\kappa_P\sigma T^4$ where σ is the Stefan-Boltzmann constant and κ_P is the Planck opacity which can be approximated by a power law model at high temperature (no radiation transfer). The application of the homothetic group leads to the new system:

$$\frac{\partial\tilde{\rho}}{\partial\tilde{t}}+\frac{a^{\delta_2}a^{\delta_3}}{a^{\delta_1}}\tilde{\vec{\nabla}}[\tilde{\rho}\tilde{\vec{v}}]=0,\quad \tilde{\rho}\frac{d}{d\tilde{t}}\tilde{\vec{v}}=-\frac{a^{\delta_2}a^{\delta_6}}{a^{\delta_1}a^{\delta_3}a^{\delta_5}}\tilde{\vec{\nabla}}\tilde{P}, \tag{3.2}$$

$$\frac{d\tilde{P}}{d\tilde{t}}-\tilde{\gamma}\frac{\tilde{P}}{\tilde{\rho}}\frac{d\tilde{\rho}}{d\tilde{t}}=-(\tilde{\gamma}-1)\left[\frac{a^{\delta_2}a^{\delta_7}}{a^{\delta_6}}\tilde{Q}_1+\frac{a^{\delta_2}a^{\delta_8}}{a^{\delta_6}}\tilde{Q}_2\right],\quad d\tilde{M}=(a^{\delta_5+(1+d)\delta_1-\delta_4})\tilde{\rho}(r,t)d\tilde{V} \tag{3.3}$$

where d is the geometrical parameter (d=0, 1, 2 in plane, cylindrical and spherical geometry, respectively) and $d/d\tilde{t}=\partial/\partial\tilde{t}+a^{\delta_2+\delta_3-\delta_1}\tilde{\vec{v}}\partial/\partial\tilde{r}$. The invariance of equations under the homothetic group enables us to extract the invariants of the group (that we have summarized in table 1), namely, $I_1 = vt/r = St$ (St: Strouhal number), $I_2 = \gamma$, $I_3 = Pt/\rho vr = Eu \times St = St/(\gamma \mathcal{M}^2)$ (Eu: Euler number and $\mathcal{M}$: Mach number), $I_4 = \mathcal{Q}_1 t/P = t/t_{\mathcal{Q}_1}$ ($t_{\mathcal{Q}_1}$: characteristic time of source phenomena), $I_5 = \mathcal{Q}_2 t/P = t/t_{\mathcal{Q}_2}$, $I_6 = M/(\rho r^{1+d})$ (mass conservation), these invariants are identical to the dimensionless numbers that usually occur in similarity studies (Castor 2007), (Blondin et al. 1990). The only difference is that, here, we have dimensionless quantities of local physical quantities contrary to ordinary dimensional considerations which give global dimensionless numbers. Thus, we make sure, by continuity, that the physical quantities are preserved. In table 2, we present scaling laws of optically thin plasmas. We see that we have four free parameters (δ_1, δ_5, δ_6 and δ_9) in the case of partial similarity. If we impose the conservation of ionization (perfect similarity), we obtain two free parameters (δ_5 and δ_6) in the case of a one source term and one free paramater (δ_5) in the case of two sources terms. In order to illustrate the rescaling of astrophysical objects or phenomena, we suggest to examine the problem of radiating shock in cataclysmic variables (Chevalier & Imamura 1982). In this case the post shock matter is cooled by bremsstrahlung process ($Q_1 \propto \rho^2 T^{1/2}$) and cyclotronic process ($Q_2 \propto \rho^{0.15} T^{2.5}$). Since Rankine-Hugoniot relations are invariants, we make sure that on the shock front we have a homothetic situation. According to the calculation reported in table 2 we see that we have one free parameter (δ_5) to rescale this problem and we can explore the properties of these structures. Thus, we can study the Chevalier-Imamura instability in an experimental way.

Now if we set $\delta_1 = 1$, $\delta_5 = 0$, $\delta_6 = 2$ and $\mathcal{Q}_2 = 0$, we get the self-similar considerations by Boily & Lynden-Bell (1995). Consequently, we can see that with the same formalism, we study similarity properties as well as scaling laws and we include all the considerations etablished for this type of plasmas.

Table 2. *Scaling laws of optically thin plasmas. Second column corresponds to general scaling laws for power laws model of source functions. We propose rescaling of plane radiating shock (d=0) in cataclysmic variables. This shock is characterized by a bremsstrahlung dominanted cooling zone [$Q_1 \propto \rho^2 T^{1/2}$] which can be Chevalier-Imamura unstable and the cyclotronic cooling zone [$Q_2 \propto \rho^{0.15} T^{2.5}$]: BC (bremsstrahlung cooling), BC+CC (bremsstrahlung cooling + cyclotron cooling).*

physical ratio	ratio (scaling factor)	BC	BC + CC
$r/\tilde{r}$	a^{δ_1}	$a^{\delta_6 - 2\delta_5}$	$a^{-3\delta_5/40}$
$\rho/\tilde{\rho}$	a^{δ_5}	a^{δ_5}	a^{δ_5}
$P/\tilde{P}$	a^{δ_6}	a^{δ_6}	$a^{77\delta_5/40}$
$t/\tilde{t}$	$a^{\delta_1 + (\delta_5 - \delta_6)/2}$	$a^{(\delta_6 - 3\delta_5)/2}$	$a^{-43\delta_5/80}$
$v/\tilde{v}$	$a^{(\delta_6 - \delta_5)/2}$	$a^{(\delta_6 - \delta_5)/2}$	$a^{37\delta_5/80}$
$T/\tilde{T}$	$a^{(\delta_6 - \delta_9 - \mu\delta_5)/\nu}$	$a^{\delta_6 - \delta_5}$	$a^{37\delta_5/40}$
$M/\tilde{M}$	$a^{\delta_5 + (1+d)\delta_1}$	$a^{\delta_6 - \delta_5}$	$a^{37\delta_5/40}$
$\alpha/\tilde{\alpha}$	$a^{\delta_6 - \gamma\delta_5}$	$a^{\delta_6 - \gamma\delta_5}$	$a^{(77/40 - \gamma)\delta_5}$
$\mathcal{Q}_{0,1}/\tilde{\mathcal{Q}}_{0,1}$	$a^{(3/2 - n_1)\delta_6 - (m_1 + 1/2)\delta_5 - (l_1 + 1)\delta_1}$	1	1
$\mathcal{Q}_{0,2}/\tilde{\mathcal{Q}}_{0,2}$	$a^{(3/2 - n_2)\delta_6 - (m_2 + 1/2)\delta_5 - (l_2 + 1)\delta_1}$	0	1

4 Conclusion

In this paper we have examined the scaling laws problem in radiating hydrodynamics. We obtain with the same formalism the similarity properties of the considered fluids as well as the self-similar behavior that the fluid can present in its evolution. As an example, we have studied the radiating shock in cataclysmic variables. We have demonstrated that we have one free parameter to rescale this astrophysical phenomena and consequently it is possible to reproduce it in laboratory.

References

Birkhoff, G. 1950 *Hydrodynamics: a Study in Logic, Fact, and Similitude* **Princeton University**

Blondin, J. M., Fryxell, B. A., & Konigl, A. 1990 *ApJ* **360** 370

Boily, C. M. & Lynden-Bell, D. 1995 *Mon. Not. R. Astron. Soc.* **276** 133

Castor, J. 2007 *Astrophys. Space Sci.* **307** 207

Chevalier, R. A., & Imamura, J. N. 1982 *ApJ* **261** 543
Chevalier, R. A., & Blondin, J. M. 1995 *ApJ* **444** 312
Cropper, M. 1990 *Space Science Reviews* **54** 195
Drake, R. P. 2006 *High-Energy-Density Physics: Fundamentals* **Springer-Verlag**
Falize, E., Bouquet, S., & Michaut, C. 2007 *Journal of Physics: Conference Series* IFSA Conference **submitted**
Remington, B. A., Ryutov, D. D., & Drake R. P. 2006 *Rev. Mod. Phys.* **78** 755
Ryutov, D. D., et al. 1999 *Astrophys. J.* **518** 821
Ryutov, D. D., Drake R. P., & Remington B. A. 2000 *Astrophys. J. Suppl. S.* **127** 465
Ryutov, D. D. et al. 2001 *Phys. Plasmas* **8** 1804