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MAGNETOHYDRODYNAMIC TURBULENCE

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Abstract. A short review is given on MHD turbulence for strong and wave turbulence, and for incompressible as well as compressible fluids. The role of anisotropy *versus* isotropy is discussed and important issues are raised.

1 Turbulence in Navier-Stokes fluids

Turbulence is one of the main problems in theoretical physics. For that reason, any exact results appear almost as a miracle. In his third 1941 turbulence paper, Kolmogorov found that an exact and nontrivial relation may be derived from Navier-Stokes equations – which can be seen as the archetype equations for describing turbulence – for the third-order longitudinal structure function (Kolmogorov, 1941). Because of the rarity of such results, the Kolmogorov's four-fifth's law is considered as one of the most important results in turbulence (Frisch, 1995). Basically, the four-fifth's theorem makes the following link between a two-point measurement, separated by a distance $\mathbf{r}$, and the distance itself (in 3D):

$$-\frac{4}{5}\varepsilon^v r = \langle (v'_\parallel - v_\parallel)^3 \rangle \, , \tag{1.1}$$

where $\langle\rangle$ denotes an ensemble average, the parallel direction $\parallel$ is the one along the vector separation $\mathbf{r}$, v is the velocity and ε^v is the mean (kinetic) energy dissipation rate per unit mass. To obtain this exact result the assumptions of homogeneity and isotropy are made (Batchelor, 1953). The former assumption is satisfied as long as we are at the heart of the fluid (far from the boundaries) and the latter is also satisfied if no external agent (like, for example, rotation or stratification) are present. Additionally, we need to consider the long time limit for which a stationary state is reached with a finite ε^v and we take the infinite Reynolds number limit ($\nu \to 0$) for which the mean energy dissipation rate per unit mass tends to a finite positive limit. Therefore, the exact prediction is valid, at first order, in a wide inertial range. This prediction is well supported by the experimental data (Frisch, 1995). Note that this type of law has been extended by Yaglom (Yaglom, 1949) to scalar passively advected (by still an incompressible fluid), such as the temperature or a pollutant in the atmosphere.

The four-fifth's law is a fundamental result used to develop scaling law models like the famous – but not exact – 5/3-Kolmogorov spectrum. For higher-order correlation functions, significant deviation from a linear law deduced directly from the four-fifth's law is found in experiments and direct numerical simulations: this deviation which has led to the development of several models is called intermittency.

2 Incompressible MHD Turbulence

2.1 Homogeneity and isotropy assumptions

For astrophysical fluids, the Navier-Stokes description is a rather poor model and we generally prefer to use the magnetohydrodynamics approximation which describes quite well the large-scale plasma dynamics. The question of the existence of an exact relation between a two-point measurement, separated by a distance $\mathbf{r}$, and the distance itself is naturally addressed. A positive answer was given by Politano & Pouquet only in 1998 (see

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also Chandrasekhar (1951)) for incompressible MHD turbulence. The addition in the analysis of the magnetic field and its coupling with the velocity field renders the problem more difficult and, in practice, we are dealing with a couple of equations. In this case, the possible formulation in 3D is:

$$-\frac{4}{3}\varepsilon^{\pm}r = \langle (z_{\parallel}'^{-} - z_{\parallel}^{-}) \sum_i (z_i'^{+} - z_i^{+})^2 \rangle \,, \tag{2.1}$$

where the parallel direction $\parallel$ is still the one along the vector separation $\mathbf{r}$, $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$ is the Elsässer fields (with $\mathbf{b}$ normalized to a velocity field) and $\varepsilon^{\pm}$ is the mean energy dissipation rate per unit mass associated to the Elsässer energies. To obtain these exact results the assumptions of homogeneity and isotropy are still made, and we also consider the long time limit for which a stationary state is reached with a finite $\varepsilon^{\pm}$ and we take the infinite (magnetic) Reynolds number limit ($\nu \to 0$ and $\eta \to 0$) for which the mean energy dissipation rates per unit mass tend to a finite positive limit. Therefore, the exact prediction is again valid, at first order, in a wide inertial range. Some comments have to be made on these 4/3's law. First, we do not really make a distinction between the viscosity ν and the magnetic resistivity η in the treatment which means that we basically assume a unit magnetic Prandtl number. Second, the isotropy assumption, which mainly appears in the development of the kinematics (Batchelor, 1953), is stronger for magnetized fluids since most of the situations found in astrophysics is far from isotropy. A good example is the solar wind where *in situ* measurements demonstrate the anisotropic nature of turbulence. Nevertheless, the use the 4/3's law gives interesting results (Sorriso-Valvo, 2007; MacBride et al., 2008). Another example is given by the Sun where many thin coronal loops are well observed which are considered as a signature of anisotropy (see *e.g.* Bigot et al., 2008). Note finally that an extension of this 4/3's theorem to 3D Hall-MHD has been obtained recently (Galtier, 2008) which provides a relevant tool to investigate the non-linear nature of the high frequency magnetic field fluctuations in the solar wind (see, *e.g.* , Goldstein et al., 1994; Markovskii et al., 2006; Galtier & Buchlin, 2007).

The four-third's law is a fundamental result which may be used to develop scaling law models for different-order correlation functions. However, the situation is not as clear as for neutral fluids since there are two time-scales in MHD: the eddy-turnover time and the Alfvén time. The former time is associated to the transfer time for Navier-Stokes turbulence; the latter time has to be seen as the time of interaction between two counter-propagating Alfvén wave packets (see Fig.1) During a collision, there is a deformation of the wave packets in such

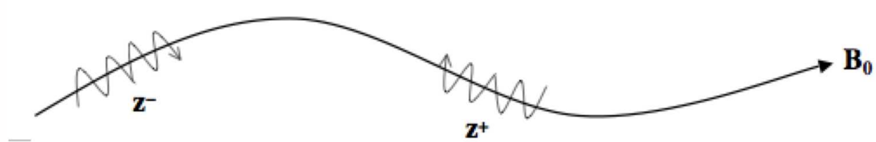

Fig. 1. Alfvén wave packets propagating along a magnetic field line.

a way that energy is transferred mainly at smaller scales. The multiplicity of collisions leads to the formation of a well extended power law energy spectrum whose index lies between $-5/3$ (Kolmogorov's prediction) or $-3/2$ (Iroshnikov-Kraichnan's (1964-1965) prediction). The precise value is still the subject of many discussions.

2.2 Beyond isotropy

The weakness of the Iroshnikov-Kraichnan (IK) phenomenology is the apparent contradiction between the presence of Alfvén waves and the absence of an external uniform magnetic field. The external field is supposed to be played by the large-scale magnetic field but its main effect, *i.e.* anisotropy, is not included in the description. One of the most important difference between neutral and magnetized fluids is the possibility in the latter case to have a large-scale field which cannot be removed by a galilean transform. This large-scale component corresponds to a large-scale magnetic field $\mathbf{B_0}$ (see Fig. 1). The role of a uniform magnetic field has been widely discussed in the literature and, in particular, during the last two decades (see, *e.g.* , Montgomery & Turner, 1981; Shebalin et al., 1983; Ng & Bhattacharjee, 1996; Verma, 2004). At strong $\mathbf{B_0}$ intensity, one of the most clearly established results is the bi-dimensionalization of MHD turbulent flows with a strong reduction of nonlinear transfers along $\mathbf{B_0}$. In the early eighties, it was shown that a strong B_0 leads to anisotropic turbulence with an energy concentration near the plane $\mathbf{k}\cdot\mathbf{B_0}=0$, a result confirmed later on by direct numerical simulations in two and three space dimensions. From an observational point of view, we have also several evidences that astrophysical (and laboratory) plasmas are mostly in anisotropic states like in the solar wind (Bruno & Carbone, 2005) or in the interstellar medium (Elmegreen & Scalo, 2004; Scalo & Elmergreen, 2004).

The effects of a strong uniform magnetic field may be handled through an analysis of resonant triadic interactions (Shebalin et al., 1983) between the wavevectors $(\mathbf{k},\mathbf{p},\mathbf{q})$ which satisfy the relation $\mathbf{k}=\mathbf{p}+\mathbf{q}$, whereas the associated wave frequencies satisfy, for example, $\omega(\mathbf{k})=\omega(\mathbf{p})-\omega(\mathbf{q})$. The Alfvén frequency is $\omega(\mathbf{k})=\mathbf{k}\cdot\mathbf{B_0}=k_\parallel B_0$, where $\parallel$ defines the direction along $\mathbf{B_0}$ ($\perp$ will be the perpendicular direction to $\mathbf{B_0}$). The solution of these three-wave resonant conditions directly gives, $q_\parallel=0$, which implies a spectral transfer only in the perpendicular direction. For a strength of B_0 well above the *r.m.s.* level of the kinetic and magnetic fluctuations, the nonlinear interactions of Alfvén waves may dominate the dynamics of the MHD flow leading to the regime of (weak) wave turbulence where the energy transfer, stemming from three-wave resonant interactions, can only increase the perpendicular component of the wavevectors, while the nonlinear transfers is completely inhibited along $\mathbf{B_0}$ (Galtier et al., 2000).

Another important issue discussed in the literature is the relationship between perpendicular and parallel scales in anisotropic MHD turbulence (see Higdon, 1984; Goldreich & Sridhar, 1995; Boldyrev, 2006). In order to take into account the anisotropy, Goldreich & Shridar (1995) proposed a heuristic model based on a critical balance between linear wave periods and nonlinear turnover time scales, respectively $\tau_A\sim\ell_\parallel/B_0$ and $\tau_{NL}\sim\ell_\perp/u_\ell$ (where $\ell_\parallel$ and $\ell_\perp$ are the typical length scales parallel and perpendicular to $\mathbf{B_0}$), with $\tau_A=\tau_{NL}$ at all inertial scales. Following the Kolmogorov arguments, one ends up with a $E(k_\perp,k_\parallel)\sim k_\perp^{-5/3}$ energy spectrum (where $\mathbf{k}\equiv(\mathbf{k}_\perp,k_\parallel)$ and $k_\perp\equiv|\mathbf{k}_\perp|$) with the anisotropic scaling law

$$k_\parallel\sim k_\perp^{2/3}\,. \tag{2.2}$$

A generalization of this result has been proposed recently (Galtier et al., 2005) in an attempt to model MHD flows in both the weak and strong turbulent regimes, as well as in the transition between them. In this heuristic model, the time-scale ratio $\chi=\tau_A/\tau_{NL}$ is supposed to be constant at all scales but not necessarily equal to unity. The relaxation of this constraint enables to still recover the anisotropic scaling law (2.2) and to find a universal prediction for the total energy spectrum $E(k_\perp,k_\parallel)\sim k_\perp^{-\alpha}k_\parallel^{-\beta}$, with $3\alpha+2\beta=7$. According to direct numerical simulations (see, *e.g.* Cho et al., 2000; Maron & Goldreich, 2001; Shaikh & Zank, 2007), the anisotropic scaling law between parallel and perpendicular scales (2.2) seems to be a robust result and an approximately constant ratio χ, generally smaller than one, is found between the Alfvén and the nonlinear times. This sub-critical value of χ implies therefore a dynamics mainly driven by Alfvén waves interactions.

In the weak turbulence limit, the time-scale separation, $\chi\ll1$, leads to the destruction of some nonlinear terms, including the fourth-order cumulants, and only the resonance terms survive (Zakharov et al., 1992; Galtier et al., 2000) which allows to obtain a natural asymptotic closure for the wave kinetic equations. In absence of helicities and for $k_\perp\gg k_\parallel$, the dynamics is then entirely governed by shear-Alfvén waves, the pseudo-Alfvén waves being passively advected by the previous one. In the case of an axisymmetric turbulence, and in the absence of cross-correlation between velocity and magnetic field fluctuations, the exact power law solution is $E(k_\perp,k_\parallel)\sim k_\perp^{-2}f(k_\parallel)$, where f is an arbitrary function taking into account the transfer inhibition along $\mathbf{B_0}$. First evidences of this regimes by direct numerical simulations are now obtained (Perez & Boldyrev, 2008; Bigot et al., 2008).

3 Compressible MHD Turbulence

Compressible turbulence has many applications in astrophysics and, in particular, in the interstellar medium where the plasma is thought to be highly compressible. For example, radio wave scintillation observations reveal a nearly Kolmogorov spectrum of density fluctuations in the ionized interstellar medium. Unfortunately, the previous exact results – the 4/5's law – has no equivalent even for compressible neutral fluids. It is therefore more difficult to predict the scaling-laws for intermittency although astrophysical data are available (see *e.g.* Burlaga, 1991). Under some restricted conditions ($\beta\ll1$), wave turbulence may apply to compressible MHD and give some exact results (Kuznetsov, 2001, Chandran, 2005). It is shown that three-wave interactions transfer energy to high-frequency fast waves and, to a lesser extend, high-frequency Alfvén waves with a strong anisotropy. In strong turbulence, one has also to make the distinction between Alfvén waves, fast and slow magneto-acoustic waves. In this regime, direct numerical simulations are the best way to extract any scaling laws (see *e.g.* Kowal et al., 2007). Basically, a different behavior is found for sub and supersonic flows with in the latter case a strong dependence in the sonic Mach number.

4 Conclusion

Many questions are still opened in astrophysical turbulence but super-computers can now reveal some new features which, conjointly with theoretical efforts, will certainly help to deeper understand the complex dynamics of MHD fluids and , in particular, in the compressible case.

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