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EXTENSION OF THE KOLMOGOROV 4/5'S THEOREM TO HALL-MHD WITH AN APPLICATION TO THE SOLAR WIND

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Abstract. I present an extension of the Kolmogorov 4/5's theorem to 3D Hall-MHD in the case of an homogeneous and isotropic turbulence. The multi-scale law found provides a relevant tool to investigate the non-linear nature of the high frequency magnetic field fluctuations in the solar wind or, more generally, in any plasma where the Hall effect is important.

1 Introduction

Turbulence remains one of the last great unsolved problem in classical physics which has evaded physical understanding and systematic description for many decades. For that reason, any exact results appear almost as a miracle. In his third 1941 turbulence paper, Kolmogorov found that an exact and nontrivial relation may be derived from Navier-Stokes equations – which can be seen as the archetype equations for describing turbulence – for the third-order longitudinal structure function (Kolmogorov, 1941). Because of the rarity of such results, the Kolmogorov's four-fifths law is considered as one of the most important results in turbulence (Frisch, 1995).

Very few extensions of such a result to other fluids have been made; it concerns scalar passively advected, such as the temperature or a pollutant in the atmosphere, and astrophysical magnetized fluid described in the framework of MHD (Chandrasekhar, 1951; Politano & Pouquet, 1998). The addition in the analysis of the magnetic field and its coupling with the velocity field renders the problem more difficult and, in practice, we are dealing with a couple of equations. In this paper, I present the extension of the Kolmogorov 4/5's theorem to 3D Hall-MHD (Galtier, 2008).

2 Hall MHD equations

We start our analysis with the following 3D incompressible Hall MHD equations

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P_* + \mathbf{b}\cdot\nabla\,\mathbf{b} + \nu\Delta\mathbf{v}\,, \tag{2.1}$$

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{b} = \mathbf{b}\cdot\nabla\,\mathbf{v} - d_I\nabla\times[\mathbf{J}\times\mathbf{b}] + \eta\Delta\mathbf{b}, \tag{2.2}$$

with $\nabla\cdot\mathbf{v}=0$, $\nabla\cdot\mathbf{b}=0$. The magnetic field $\mathbf{b}$ is normalized to a velocity ($\mathbf{b}\to\sqrt{\mu_0 n m_i}\,\mathbf{b}$, with m_i the ion mass and n the electron density), $\mathbf{v}$ is the plasma flow velocity, P_* is the total (magnetic plus kinetic) pressure, ν is the viscosity, η is the magnetic diffusivity and d_I is the ion inertial length ($d_I = c/\omega_{pi}$, where c is the speed of light and ω_{pi} is the ion plasma frequency); $\mathbf{J}=\nabla\times\mathbf{b}$ is the normalized current density.

3 Result

The extension of the Kolmogorov 4/5's theorem to 3D Hall-MHD is (Galtier, 2008):

$$-\frac{4}{3}\varepsilon^T r = B^{vvv}_{\|ii} + B^{vbb}_{\|ii} - 2B^{bvb}_{\|ii} + 4d_I\langle[(\mathbf{J}\times\mathbf{b})\times\mathbf{b}']_{\|}\rangle\,,$$

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where ε^T is the mean (total) energy dissipation rate per unit mass, $B_{ijk}^{\alpha\beta\gamma} = \langle(\alpha'_i - \alpha_i)(\beta'_j - \beta_j)(\gamma'_k - \gamma_k)\rangle$ is the third-order structure function combining the (i,j,k) components of the vector fields α, β and γ measured at two different points (M and M') separated by a distance $\mathbf{r}$ which defines the parallel direction. Note the use of the Einstein's notation.

To find this exact result, we assume homogeneity and isotropy (Batchelor, 1953). Furthemore, we consider the long time limit for which a stationary state is reached with a finite ε^T; we take the infinite (magnetic) Reynolds number limit ($\nu \to 0$ and $\eta \to 0$) for which the mean energy dissipation rate per unit mass tends to a finite positive limit. This result is thus valid at first order in the inertial range.

4 Discussion

The most remarkable aspect of this exact law is that it does not only provide a linear scaling for the third-order correlation tensors within the inertial range of length scales, but it also fixes the value of the numerical factor appearing in front of the scaling relations. Another important remark is about the fields used to build the third-order correlation tensors. Indeed, the convenient variables are not only the velocity and magnetic field components but also the current density components.

The exact result presented here provides a better theoretical understanding of Hall MHD flows. It shows that the scaling relation does not change its power dependence in the separation r at small-scales if the statistical correlation tensor used is modified. The interesting point to note is the compatibility with previous heuristic and numerical results (Biskamp et al., 1996). Indeed, a simple dimensional analysis gives the relations $r \sim b^3$ for large-scales, and $r^2 \sim b^3$ for small-scales (since $J \sim b/r$), which give respectively the magnetic energy spectrum $E \sim k^{-5/3}$ and $E \sim k^{-7/3}$. Therefore and contrary to the appearance, the exact result found may provide a double scaling relation.

These multi-scale law provides a relevant tool to investigate the non-linear nature of the high frequency magnetic field fluctuations in the solar wind whose (dissipative *vs* dispersive) origin is still controversial (Goldstein et al., 1994; Galtier, 2006). The use of multi-point data may give information about both the magnetic field and the current density which can be used to check the theoretical scaling relations. The observation of such a scaling law would be an additional evidence for the presence of a dispersive inertial range and therefore for the turbulent nature of the high frequency magnetic field fluctuations. The recent observation of the Yaglom MHD scaling law (Sorriso-Valvo et al., 2007) at low frequency provides a direct evidence for the presence of an inertial energy cascade in the solar wind. The theoretical results given here allows now to extend this type of analysis to the high frequency magnetic field fluctuations and, more generally speaking, to better understand the role of the Hall effect in astrophysics.

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